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ABSTRACT

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the Third International Congress for Mathematics Education, to promote international contacts and the exchange of scientific information in the psychology of mathematical education. The goal was to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof. This material consists of two documents. The first is a collection of approximately 70 papers classified under the following themes: (1) Concept Formation; (2) Problem Solving; (3) Language; (4) Proof; (5) Arithmetic and Algebra; (6) Ratio and Proportion; (7) Methods of Teaching; (8) Assessment; (9) Errors, (10) Discovery Learning; (11) Social Aspects; and (12) Neurophysiology. The second volume contains two of the five plenary lectures presented at the conference, a listing of the contents for the first document, and five additional conference papers. (MP)

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pme 6

Antwerp 18 - 23 July 1982

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Edited by

Alfred Vermandel

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PREFACE

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the 3rd International Congress for Mathematics Education in Karlsruhe, in order to promote international contacts and the exchange of scientific information in the psychology of mathematical education so as to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof. Conferences of the IGPME were held in Utrecht, Osnabrück, Warwick, Berkeley and Grenoble during the five subsequent years. The Sixth International Conference will take place on the campus of the University of Antwerp from 18-23 July 1982.

The scientific programme of the 6th Conference includes :

- five plenary lectures setting forth main themes in the psychology of mathematics education and of mathematical thinking.
- approximately 70 papers which will be read in parallel sessions. The texts of these papers are collected in the present volume. They have been classified by the International Programme Committee under the following themes :

A. Concept formation
B. Problem Solving
C. Language
D. Proof
E. Arithmetic and Algebra
F. Ratio and Proportion
G. Methods of teaching
H. Assessment
I. Errors
J. Discovery Learning
K. Social aspects
L. Neurophysiology

The order of the papers in these proceedings will not necessarily be identical with the order in which they are read in the congress. Complete details will be given in the full congress programme.
A table of contents at the beginning ordered along the above themes and an alphabetic name listing of the contributors with their addresses and the page reference may be used to locate a particular contribution.

A second volume containing the papers of the plenary lectures is expected to be edited by the time of the conference.

We are grateful to the contributors for their interesting presentations and their coöperation in furnishing the manuscripts promptly and in the form requested. We are grateful to the national and local sponsors of the conference mentioned on the head page. We also wish to express our gratitude to R. Skemp, H. Freudenthal, C. Comiti, R. Hershkowitz, E. Cohors-Fresenborg and F. Lowenthal for their preparatory reading of the papers and for having come to Antwerp to help us design the conference programme. Finally we thank Marleen Van Barel for her careful and speedy preparation of the printer's copy.

Alfred VERMANDEL, chair and editor
Eric SCHILLEMANS, congress director

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11

A. CONCEPT FORMATION

THE UNDERSTANDING OF ALGORITHMIC CONCEPTS ON THE BASIS OF ELEMENTARY ACTIONS

Elmar Cohors-Fresenborg, Universität Osnabrück

The study concerns thinking-processes of 13 years old students constructing and analyzing algorithms as programs for a Registermachine. The basic experimental design consisted of "guided discovery" in "multiple embodiments" (DIENES) of the tasks. Each of the 14 subjects had six individual sessions with the experimenter and worked first on a set of tasks where programs had to be constructed and then on another set where given programs had to be analyzed with regard to their function. The multiple embodiment approach was especially organized as to yield different inner representations in the sense of BRUNER.

The concept of Registermachines is a further development by RÖDDING (1968) of the ideas of a mathematical machine, which was first described by MINSKY (1961). The programming-language is defined recursively by the processes of *concatenation* (sequential application), *iteration* (controlled by a zero test in a register) from the elementary operation of *counting forward* and *backward*.

One of the most important prerequisitions of the present study has been to base the construction of algorithms (for the Registermachine) on viable mental representations of the enactive, iconic and symbolic mode. As an example of how this has been accomplished the iterative computation of addition from the successor-function shall be described.1)

As an elementary action one has to organize the sequence of taking away and putting down of matchsticks in two columns, representing the algorithm for the Registermachine:

```
Repeat until storage 2 is empty:
Take one matchstick away from storage 2.
Add one matchstick in storage 1.
```

The execution of this algorithm is drawn step by step in figure 1.

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1) Of course there is not ment the simple arithmetic skills of computation (e.g. in place value systems) but the invention of an algorithm which realizes the idea of recursive definition.

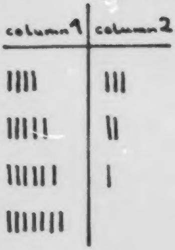

Fig. 1

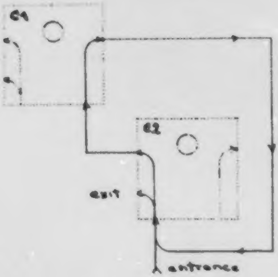

Fig. 2

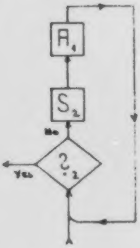

Fig. 3

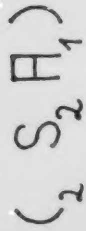

Fig. 4

In a second step this sequence of actions has to be represented as a working flow-chart. For that we had the didactical material *Dynamische Labyrinthe* (Dynamic Mazes). The repetition of taking away and adding is represented by a so-called calculation-network (fig. 2). Here the execution of the two elementary actions (taking away, adding) corresponds to the use of the left or right entrance respectively in a so-called counting-brick. The concatenation of the two elementary actions is represented by the sequential use of this calculation network. The repetition of the sequence taking away/adding until column 2 is empty is represented by a loop which is evident in the calculation network.1)

The places of mathematical interest in the network which are counting backward in counter 2, counting forward in counter 1 and the repetition of this sequence until counter 2 is zero can be represented in *symbolic notation* as the program word: $({}_2S_2A_1)$, see fig. 4.

We have constructed a model-computer "*Registermachine*" by which you can follow the execution of such a program word letter by letter while the curser is moving in the program word. A pupil learns to use the Registermachine within some minutes.

1) A calculation-network like this can be considered as a working flow-chart.

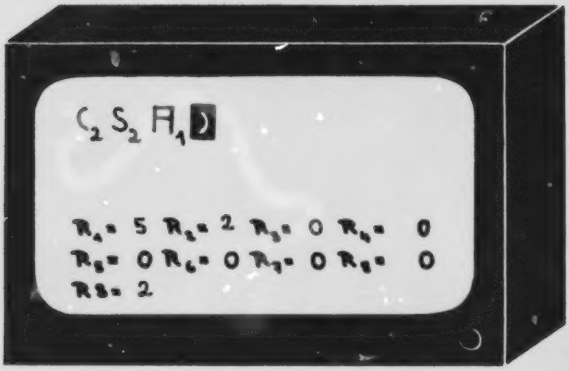
$(_2 S_2 A_1)$
R_1= 5 R_2= 2 R_3= 0 R_4= 0
R_5= 0 R_6= 0 R_7= 0 R_8= 0
RS= 2

Figure 5

The pupil can follow on the monitor (fig. 5) the movement of the curser and simultaneously the changing of the contents of the registers $R_1,\ldots,R_8$ and the stepcounter RS. It is possible to stop the Registermachine during a computation at any time, even for a correction of the program.

If one compares our design with the stages of representation in the sense of BRUNER (1966) and the comments of RESNICK/FORD (1981) on the representation of mathematical knowledge, one can see the following correspondence:
The algorithm is represented enactively, if you consider the sequences of actions with the matchsticks (fig. 1). If one remembers the going through the constructed counting network (fig. 2), it is an enactive representation, too. If you consider the figure 2, you have an iconic representation as well as by the flow-chart (see fig. 3). The symbolic notation in fig. 4 gives you a symbolic representation.

A representation of the concept of iteration/loop is of particular interest.
In the sequence of elementary actions with the bricks the concept of loop is hidden in the concious repetition of the actions. In the enactive representation of the counting network you can show, where the concept is represented during the action: going round. In the iconic representation by flow-chart you can see the loop. The pair of brackets represents the concept in the symbolic notation.

On one hand the tasks for the students consisted of writing Register-machine-programs for addition, subtraction, multiplication, division and several simple linear functions. The pupils could choose in which level of representation (acting with the matchsticks, construction of the corresponding network) they started to solve the problem. We specially put strain on the first tasks to make clear to the pupils the different levels of representation of the algorithm. During the step by step execution of e.g. their addition program by the Registermachine they had to execute the algorithm on the other levels (acting with the matchsticks, running through the calculation network).

Besides, such problems for the construction of an algorithm the pupils had to solve problems from the third lesson upward which consisted in giving the number of steps which the Registermachine needs to compute the value of the function (e.g. the step function of our addition program is $x_2 \cdot (1+1)$ and for the multiplication program mentioned below, it is $x_2 \cdot (1+3x_1+2x_1)$).These problems were taken form the lesson unit *Registermachines and functions* (COHORS-FRESENBORG et al.^{3}1982) which where used several times in mathematic lessons with pupils of 14 years of age.

Example of an analytic task

We shall explain the different examples of understanding at an analytic problem with the example of the program word

$$({}_2\, S_2\, A_1)$$

A first level of understanding consists for pupils in the explanation of the effect of such programs that they only give the step by step execution of the several commands (zero-test, counting backward, counting forward, start again) to explain the effect of the given program. For a long time some of them stay on this level. They are able to delineate this program by a corresponding action with matchsticks like a Registermachine.

On a second level they are able to describe the effect of the program in the following way. You have to count backward in the 2nd register and to count forward in the 1st register until the 2nd register is empty. Only the third level they are able to give the effect of such a program by stating the corresponding mathematical function (in this case: addition).

It is more complicated to give the effect of the program word for multiplication

$$({}_2 S_2({}_1 S_1 A_3 A_4)({}_4 S_4 A_3))$$

In this you have to recognize and to understand the two loops, put one into the other which are represented in the program word by pairs of brackets.

Summary

The experiment may be characterized as "Learning by discovery". But between the first instruction given to the subjects and the final solution a network of hints was interpolated, so that the experimental procedure was modified into "guided discovery". We intended the system of hints to be dense enough that practically every subject could be helped towards the complete solution. The actual performance of a student has been assessed by the proportion of total hints which he required to attain the solution.

One part of the tasks consisted in constructions of an algorithm and the other in the analysis of what a given program will do. So the total performance score could be splitted into an "analytical" and a "constructive" one, and the subjects could be additionally classified according to these two categories. While there are clear cases of consistent belonging to one or the other categorie, we do not yet know what the dichotomy really means psychologically.

In our experiments we found examples for all four categories:

1. students are successful in both types of tasks
2. students are not successful in both types of tasks
3. students are more successful in analytic tasks
4. students are more successful in constructive tasks

The fact, that we found examples for the last two categories shows that there doesn't exist a hierarchy between analytic an constructive. In this context we don't understand the usual hierarchy of learning sets, where construction follows analysis.

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Resnick, L.B./ Ford, W.W.	The psychology of mathematics for introduction; Hillsdale 1981
Rödding, D.	Klassen rekursiver Funktionen; in: Lecture Notes in Mathematics, Vol. 70, Berlin 1968

The set of bricks "Dynamische Labyrinthe" is produced by Beschützende Werkstatt, Industriestraße 2, D-4500 Osnabrück

The counting networks are described in:

Cohors-Fresenborg, E./ Finke, D./ Schütte, S.	Dynamic Mazes, Booklet 9; Osnabrücker Schriften zur Mathematik, Reihe U, Heft 19, Osnabrück 1979

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ETUDE D'UNE SEQUENCE DIDACTIQUE A PROPOS DE LA NOTION DE LIMITE

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This work is a part of a research about the learning of the concept of limit. It was preceded by two approaches : one through the spontaneous conceptions of the notion, before any learning ; the other through the historical development of the concept of limit, to see the main difficulties which were encountered, and how they were overcome .

We have studied a learning situation for 16-17 years pupils. We wanted to start with a situation with problems for which the notion of limit was necessary, so that the pupils get a visual approach of the notion, and that the notion works, before it has a precise mathematical definition. We wanted to see which words the pupils would use to describe the situations, and which main difficulties they would encounter : the relation with the concept of infinite; the relation between the concept of limit and monotony; the level of abstraction of the concept; the difficulty to consider what happens "at infinity" as a limit, and not as a thing in itself; the passage to the quantitative aspect .

INTRODUCTION : Le travail présenté ici est une partie d'un travail à plus long terme, concernant l'acquisition de la notion de limite. Il a été précédé par deux approches, qu'il convient de rappeler :

Tout d'abord une approche par l'étude des *conceptions spontanées* liées à la notion de limite: Un test proposé à des élèves avant tout enseignement au sujet de la limite, a permis de mettre en évidence, à travers le sens accordé aux expressions "tend vers" et " a pour limite", diverses conceptions spontanées. Les principales conceptions ainsi observées étaient : le fait de se rapprocher de ... ; le fait de se rapprocher de ... jusqu'à l'atteindre; le fait de se rapprocher de ... sans jamais l'atteindre; le fait de "ressembler à"; la notion de borne, d'obstacle infranchissable, etc... Ces conceptions spontanées, souvent attachées au vocabulaire employé pour parler de la notion de limite, constituent un terrain sur lequel se bâtira la notion, au fur et à mesure de l'enseignement. Nos tests ont montré que la notion mathématique ne prendra pas purement et simplement la place des conceptions spontanées, mais qu'il se formera un mélange, donnant lieu chez chaque élève à ce que nous appelons des *conceptions propres* .

L'autre approche précédant notre étude a été une approche historique et épistémologique: En parcourant les grandes étapes de l'élaboration du concept de limite, depuis Euclide jusqu'à la formalisation du concept au XIXème siècle, nous avons tenté de discerner dans quels champs la notion a pu se développer, quelles problématiques ont été à l'origine de progrès (le rôle de la géométrie, de la mécanique; le calcul des dérivées ...) , quels obstacles ont dû être surmontés, et de quelle façon on a pu les surmonter . Cette approche apporte un éclairage sur ce qui se passe aujourd'hui lorsqu'un élève doit acquérir la notion de limite : non pas que l'élève ait à parcourir à nouveau le chemin historique de l'élaboration de cette notion; mais il aura à surmonter dans son apprentissage certains des obstacles repérés dans l'histoire .

LA "SEQUENCE DIDACTIQUE" :

Nous avons voulu étudier une situation d'apprentissage de la notion de limite dans une classe de 1ère C (élèves de 16-17 ans). Pour cela, nous avons élaboré une séquence d'activités. Elle était destinée à des élèves n'ayant eu aucun enseignement concernant la limite. Les deux activités que nous allons décrire ici constituaient le début de l'apprentissage.

Plusieurs hypothèses nous ont guidé dans l'élaboration de cette séquence :

Plutôt que de partir d'une définition de la limite donnée d'emblée, et d'étudier sa signification et son fonctionnement à travers des exemples, des situations diverses, etc..., nous avons préféré proposer des activités permettant d'amener la notion de limite comme un outil nécessaire pour résoudre des problèmes. Cela permet de "visualiser" la notion de limite, de la faire fonctionner, avant de lui avoir donné un statut mathématique précis. A partir de l'aspect intuitif et qualitatif, nous avons ensuite voulu conduire les élèves à étudier l'aspect quantitatif, puis, à partir de là, à mettre en place une définition "rigoureuse" .

Deux questions ont été centrales dans l'élaboration de ces activités : Tout d'abord, les élèves sauraient-ils voir qu'une situation comporte la notion de limite, et quel vocabulaire emploieraient-ils pour décrire une telle situation ? Ensuite, quels obstacles essentiels rencontreraient les élèves dans leur apprentissage de la notion ?

La première séquence comporte trois activités, se situant dans des champs mathématiques différents :

- L'utilisation du cercle comme limite de polygones réguliers pour calculer le rapport des aires de deux cercles (problème emprunté à Euclide !) .
- Le calcul de la pente d'une tangente en considérant la tangente comme limite de cordes .
- L'étude de suites du type : 0.3 , 0.33 , 0.333 , ...

Les fiches destinées aux élèves étaient rédigées sans employer de vocabulaire lié à la notion de limite : nous avons souhaité qu'ils découvrent eux-mêmes qu'il s'agissait d'un problème de limite, à partir de la situation, et non à partir des mots employés .

La deuxième séquence a pour thème principal la recherche de la limite en 0 de $\frac{\sin x}{x}$. Elle comprend d'abord la mise en place de la méthode : on part du phénomène géométrique: la tangente en $x=0$ à la courbe $y=\sin x$ s'obtient comme limite de cordes. On transpose dans le champ numérique : la pente de la tangente est la limite des pentes des cordes. On met en place ensuite le calcul effectif : on calcule $\frac{\sin x}{x}$ pour des x de plus en plus petits, d'abord avec un chiffre après la virgule, puis avec deux, puis trois, etc... On obtient un tableau à double entrée. On contate que $\frac{\sin x}{x}$ se rapproche de 1 lorsque x se rapproche de 0. Les élèves sont invités à formuler ce phénomène à leur façon. Le tableau permet aussi d'introduire l'aspect quantitatif de la notion de limite, et de s'acheminer vers la définition formelle.

Cette deuxième séquence se présente sous la forme d'une fiche; les élèves travaillaient par groupes de trois ou quatre ; plusieurs de ces groupes ont été enregistrés pendant leur travail.

Les principales observations que ces séquences nous ont permis de faire sont les suivantes :

L'acquisition de la notion de limite par les élèves est liée à la perception qu'ils ont de l'infini. Le fait que la limite soit atteinte ou pas, l'éventuelle existence d'un "dernier nombre entier" avant l'infini, la nature des "infiniment petits", sont des obstacles liés à l'infini, dont le rôle est central dans l'apprentissage du concept de limite.

Un autre obstacle à franchir est le lien étroit que les élèves font entre le fait de se rapprocher de, et la monotonie.

La notion de limite n'est pas pour les élèves de même nature que les mathématiques qu'ils ont faites jusqu'alors. Il ne suffit pas de mener à bien un calcul, ou de faire un bel enchainement logique. Il y a un degré d'abstraction supplémentaire. Cela fait que la notion de limite apparait au début comme "peu rigoureuse", comme échappant au champ des mathématiques. Cela se traduit par le fait que les élèves cherchent à traiter les problèmes de limite de façon "statique" : ils regardent tout à fait séparément ce qui se passe "pour n fini", ou "pour $x \neq 0$", d'une part, et d'autre part "pour n infini" ou "pour $x=0$", sans chercher à voir le lien qui peut exister : l'idée du passage à la limite n'est pas présente.

Enfin, il faut signaler les obstacles liés à l'aspect quantitatif et à la formalisation : le passage de la notion de valeur approchée à celle de valeur approchée "aussi bien qu'on veut", la notion de condition suffisante, etc...

Il faut noter que la notion de limite comporte de multiples aspects: selon la façon dont on veut la faire fonctionner, il faut utiliser tel ou tel aspect, et par conséquent avoir surmonté tels ou tels obstacles. Il est intéressant d'examiner dans l'enseignement quels aspects de la notion de limite sont privilégiés (par exemple: l'étude des formes indéterminées), et quels obstacles leur sont liés.

Some Aspects of the Function Concept in College Students and Junior High School Teachers(*)

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§ 1. Introduction

It is a common habit in Mathematics to introduce new notions by means of verbal definitions which characterize them definitely and ultimately. Some people even consider definitions crucial to the teaching of Mathematics. That is probably because definitions have a crucial role in the Formalist description of Mathematics. However, a cliam was made (Vinner, 1980; Vinner and Hershkowitz, 1980) that in mathematical behavior what mainly matters is the Concept Image and not the Concept Definition. The concept image consists of all mental pictures and all properties which are associated with a given notion in somebody's mind. The concept image is determined by specific examples and by the accumulated experience with the notion. Quite often there is a gap or even a conflict between the student's concept image and the concept definition as taught by the teacher.

This study examines some aspects of the notion of function in college students and junior high school teachers. A previous study (Vinner, 1980) examined similar aspects in high school students to whom functions were introduced in the Dirichlet-Bourbaki approach: a function is any correspondence between two sets (the domain and the range) which assigns to every element in the domain exactly one element in the range. It was found that only 57% of the students gave one version or another of the Dirichlet-Bourbaki definition when asked about the notion of function and only 34% of these really acted according to this definition when asked various questions about functions.

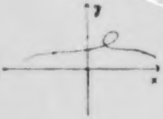
(*) This research was done while the authors were research fellows at the Weizmann Institute of Science, Rehovot, Israel.

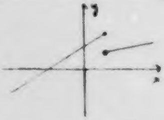
It is not intended in this study to examine concept images versus concept definitions since it is impossible to find out how functions were taught to our students. What we try here is to reveal concept images of students a few years after they learned the notion of function in high school. We examined our students' concept of function before this concept was discussed in their present classes.

§ 2. The Notion of Function and the Questionnaire

The Dirichlet-Bourbaki definition of a function came to give recognition as functions to many correspondences which were not recognized as functions by previous generations of Mathematicians (see Malik, 1981). Among them are incontinuous functions, functions defined on a split domain, functions with a finite number of "exceptions" and functions defined by means of a graph. We designed a questionnaire to examine whether various correspondences are considered as functions by the students. In this paper, because of space problem, we will discuss only 5 out of its 7 items.

Does there exist a function the graph of which is:

1.


2.


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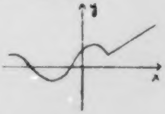

4. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1.
5. What is a function in your opinion?

In each of the questions 1 to 4 the student had to choose between "yes", "no" and "I do not know" and to explain his choice. In question 5 we did not ask about the *definition* of function because we wanted also students who had forgotten the definition to come up with something that at least reflected their concept image.

§ 3. The Sample

The questionnaire was administered to several groups of first and second year college students whose main subjects of study were Mathematics, Physics, Chemistry, Biology, Economics, Agriculture, Technological Education and Industrial Design. It was also administered to junior high school Mathematics Teachers. When analysing the data we retained only those questionnaires which contained either a definition of the function concept (Question 5) plus at least one explanation to another question or, in case the definition was missing, at least two explanations to questions out of 1 to 4. Thus we were left with 271 students and 36 teachers. They were classified into five groups according to the level of their present Mathematics training: low level (33 students), intermediate (67), high (113), Math. major students (58) and teachers (36).

We also classified the students according to time and level of their previous mathematical training but no significant differences were found on the ground of these classifications.

§ 4. The Results

The first analysis we made is related to the concept definition of function. The various definitions (the answers to question 5) were categorized into 6 categories (which were a refinement of the categorization in Vinner, 1980):

I. The function is *any* correspondence between two sets that assigns to every element in the first set exactly one element in the second set (the Dirichlet-Bourbaki definition).

II. The function is a *dependence* relation between two variables (y depends on x).

III. The function is a *rule* of correspondence (this conception eliminates the possibility of *arbitrary* correspondences).

IV. The function is a manipulation or an operation (one acts on a given number, generally by means of algebraic operations, in order to get its image).

V. The function is a formula, an algebraic term or an equation.

VI. The function is identified, probably in a meaningless way, with one of its visual or symbolic representations (the graph, the symbols "y = f(x)", etc.).

Table I shows the distribution of the concept definition categories in the five groups mentioned at the beginning of this section.

Table I - Distribution of concept definitions in the subgroups (in percents)

Mathematical level Categories	Low	Intermediate	High	Math-majors	Teachers
I	6%	18%	15%	45%	70%
II	37%	27%	32%	21%	8%
III	12%	13%	8%	12%	8%
IV	6%	2%	6%	5%	3%
V	18%	19%	7%	5%	0%
VI	12%	9%	10%	5%	3%
Others	9%	12%	22%	7%	8%
	100%	100%	100%	100%	100%

A χ^2-test shows that the differences between the subgroups are significant, $p < 0.0001$. The table shows that the percentage of the Dirichlet-Bourbaki definition increases with the level. It is specially high in the teachers. However, the difference between the 4 student groups are also significant, $p < 0.003$.

Table II shows the distribution of correct answers to questions 1, 2, 3 and 4 in the above 5 groups.

Table II - Distribution of correct answers in the sub-groups

Mathematical level Question	Low	Intermediate	High	Math majors	Teachers
1	55%	66%	64%	74%	97%
2	27%	48%	67%	86%	94%
3	36%	40%	53%	72%	94%
4	9%	22%	50%	60%	75%

Again, the differences between the 5 subgroups are significant for each one of the question ($p < 0.0007$ for question 1 and $p < 0.0001$ for questions 2-4). When excluding the teachers from the sample the differences between the four student groups remain significant ($p < 0.0008$) for questions 2-4, but not for question 1.

An additional analysis we made concerned the explanations for accepting or rejecting the existence of functions as mentioned in the questionnaire. Here the concept image played very often a crucial role. In each question different aspects of the concept image were the ground for accepting or rejecting the existence of the function. Sometimes the same aspect was the ground for rejection for some students and the ground for acceptance for other students. For instance, in some explanations to *negative* answers to question 2 it was said that the graph is discontinuous and therefore it cannot be a graph of a function. In some explanations to *positive* answers to the same question it was said that discontinuous functions are legitimate members of the "function family".

We will list here some of the major aspects of the function notion that played a crucial role in the explanations Table III will show their distribution in the answers to questions 1-4 for the entire sample.

1. One Value (OV): If a correspondence assigns exactly one value to every element in its domain then it is a function. If not - then it is not.

2. Discontinuity (D): The graph is discontinuous or changes its "character" (2 different straight lines).

3. Split Domain (SD): The domain of the function splits or the graph changes its "character" (a straight line and a curved line). (Also this aspect, as 2 above, was a ground for rejection and acceptance.)

4. Exceptional Point (EP): There is one point of exception (also a ground for rejection and acceptance).

Table III - Distribution of explanations
to questions 1-4, in the entire sample

Question	Answer	Number of Answers	Aspect					
			OV	D	SD	EP	Other	None
1	yes	46	0%	0%	4%	2%	66%	28%
	no	216	59%	0%	1%	4%	19%	17%
2	yes	203	33%	18%	16%	0%	18%	15%
	no	87	0%	40%	41%	0%	5%	14%
3	yes	176	23%	1%	28%	1%	13%	34%
	no	103	1%	0%	77%	3%	5%	14%
4	yes	138	10%	1%	35%	3%	27%	24%
	no	100	5%	1%	2%	38%	35%	19%

Similar tables were constructed for each of the five subgroups. They will be discussed elsewhere.

It was also found that concept definitions remained very often inactive in decision making moments when (sometimes wrong) concept images took over. This resulted sometimes in an answer contradictory to the concept definition. This, however, happened less often than in the previous study (Vinner, 1980). This is probably because relatively long time had passed since the students learned the notion of function. During this time the concept image and the concept definition adjusted themselves to each other.

References:

Malik, M. A., Historical and Pedagogical Aspects of the Definition of Function, International Journal of Mathematics Education in Science and Technology, 11, 489-492.

Vinner, S., Concept Image, Concept Definition and the Notion of Function, 1980 (to appear).

Vinner, S. and Hershkowitz, R., Concept Images and Common Cognitive Paths in the Development of Some Simple Geometrical Concepts, Proceeding of the Fourth Conference of IGPME, Edited by Robert Karplus, 1980, pp. 177-184.

BASIC GEOMETRIC CONCEPTS - DEFINITIONS AND IMAGES

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Shlomo Vinner, Hebrew University, Jerusalem, Israel

1. INTRODUCTION

The notion of concept image was introduced in Vinner (1980) and in Vinner & Hershkowitz (1980), as the collection of mental images that an individual has concerning a given concept. The role of the concept image in the performance of mathematical tasks, such as identification and construction, was discussed. An individual's concept image can be partial, that is, it does not contain all the aspects of the concept which fit the concept definition, or it can be complete. It can also be incorrect, in that items are incorrectly associated with the concept.

The notion of Common Cognitive Path (CCP) was also introduced. For example, suppose that A_i, $i = 1, 2, 3, 4$ are 4 different aspects of a given concept. Further, consider the condition: *knowledge of A_i implies knowledge of A_k for every $k < i$.* If all or "almost all" of a given population satisfy this condition, we say that $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4$ is a Common Cognitive Path (CCP) for the given population. To determine whether some cognitive path is a CCP requires an appropriate statistical test, in order to determine what might be meant by "almost all" above. After some thought (cf. Vinner & Hershkowitz, 1980), we came to the conclusion that the most suitable test is the Guttman scale analysis, [illegible]).

In this paper we shall describe some further applications of these concepts and we shall also examine the effect of concept definition on concept image.

2. RESEARCH AIMS, THE SAMPLE AND THE INSTRUMENT

The aims of the research are:

a) To analyze the student concept image of basic geometrical concepts as occuring in identification and construction tasks.
b) To identify hierarchical structures for basic geometric concept images in the population (CCP).
c) To analyze the effect of concept definition on the concept image for familiar basic geometrical concepts, in identification and construction tasks.
d) To analyze the process of creation of new geometrical concepts defined verbally, by means of identification and construction tasks.

In order to achieve these aims we administered two parallel versions of a questionnaire to 189 students age 11-14 (grades 6 to 8). The two versions were administered to students randomly selected, in equal numbers in each class. In the questionnaire, the student was asked to identify or to construct basic geometrical concepts which were in his experience from preschool and school years. For example, identification of angles, triangles, quadrilaterals and polygons; construction of altitude in a triangle and diagonals in a polygon. In version I of the questionnaire, a verbal concept definition was given before the student was asked to identify or to construct. In version II the same task was given without the verbal definition of the concept. In addition to the above, a verbal definition of two new concepts, specially created for the study, were given in the two versions. In version I the student was asked to identify those forms, among others, that fitted the definition, whereas in version II he was asked to construct the defined new concept.

3. RESULTS AND DISCUSSION

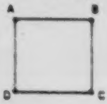
We will not bring the complete analysis of the questionnaire, but sufficient examples to show how these results throw light on the research questions.

3.1. *RESULTS RELEVANT TO THE ANALYSIS OF CONCEPT IMAGE AND ITS CCP.*

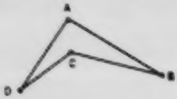
(Research aims a and b.)

Example: (This item is the same in the two versions).

Among the following shapes indicate those which are quadrilaterals:



i



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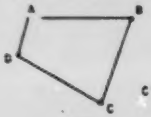

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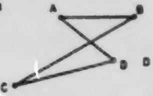

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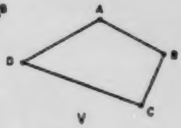


v

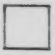
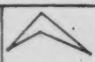
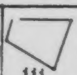
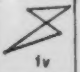
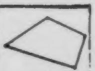
For each shape that is not a quadrilateral explain why.

Only 4 students out of the 189 did not respond. The results are given in Table 1, from which we can see, for example, that the concept image of almost all students includes square but only 22% include shape (iv) in their concept image. We also note that almost all students gave some reason for rejecting a particular shape, and some of them gave more than one.

Guttman scale analysis of student responses shows that 92% of the student have the same "cognitive path" for those items which are quadrilaterals. This path, arranged from the easiest to the most difficult, is:
i ⟶ v ⟶ ii ⟶ iv. Thus, for example, any student who recognizes ii as quadrilateral also recognizes v and i. The coefficient of reproducibility

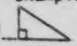
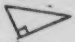
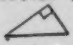
(the ratio of number of correct responses to the sum of the responses) on the Guttman scale is 0.9338, signifying a high level of validity for this structure. The subdivision of the 92% of students who have this CCP is as follows: 20% of the students have a concept image that includes all the quadrillaterals (i, v, ii and iv), 42% have shapes i, v, and ii, 13% the shapes i and v only, and the remaining 17% include only the square in their concept image.

Table 1: Analysis of items responses

		 i	 ii	 iii	 iv	 v
	marked as quad.(%)	95%	67%	14%	22%	78%
	not marked	5	59	158	143	37
	rejected without reasoning	-	12	13	14	10
REASONS	not closed			108	-	
	sides intersect			-	9	
	does not possess some attribute of the quad.			21	56	
	does not fit the definition of a square*		27	21	22	22
	identified as a shape of another name and therefore not a quad.	4	-	-	35	-
	no mathematical reasoning		8	-	11	3
	other	1	14	8	9	6
	no. of reasons	5	49	161	142	31

3.2. *THE EFFECT OF THE PRESENCE OF THE DEFINITION OF A FAMILIAR CONCEPT ON ITS CONCEPT IMAGE* (Research aim c)

We found that in *identification* tasks concept definition has almost no influence on concept image in the context of a familiar concept. In each of the identification tasks which were given (identification of right angle, obtuse angle, right-angled triangle, isosceles triangle and polygon) there is no systematic difference between the two versions. Most of the identification tasks were very easy and maybe we have here a ceiling effect. However the results of the two versions show that there are two factors which have a considerable influence on the student concept image:

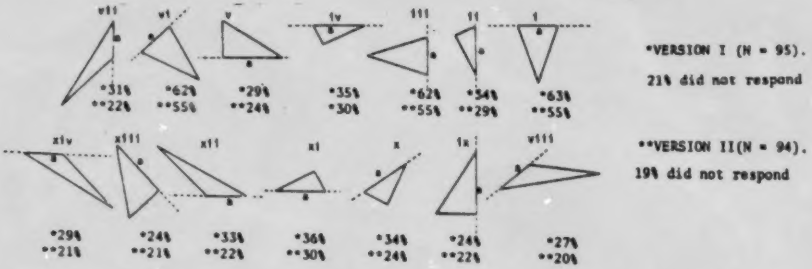
1) The existence of a "distracting" property in a geometrical shape inhibits its identification. For example, if the triangle is right-angled the student has difficulty in identifying it as isosceles.
2) An orientation factor; for example, the rotation of a right-angled triangle from a "canonical" position  to a non-standard position  inhibits identification (from 85% to 60%). An additional rotation to this position  , reduces the success to a mere 27%. (Similar results were reported by Vinner and Hershkowitz, 1980).

In contrast to the situation in the identification tasks, in *construction* tasks the concept definition seems to have influence.

Example: (Version I: with definition.)

Definition: In a triangle, the perpendicular from the vertex opposite side a to side a or its extension, is the altitude to side a.

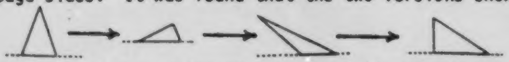
Draw the altitude to side a in each of the following triangles.



In version 2 the same item appears but without the definition. Among the 14 triangles in this item there are only 4 that are essentially different; each of these appears in 3 or 4 positions.

The success rate for each triangle in the two versions is indicated in the diagram above. We see that students who had the definition, scored systematically better. It is also clear that the kind of triangle considerably influences the success rate. For example, the success in version I dropped from 60%, in the case of an isosceles triangle, to 24% for a right-angled triangle.

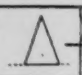
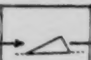
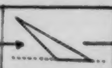
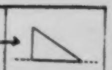
In this item it is interesting to note that the orientation factor, has almost no effect. Hence, we performed the Guttman scale analysis on one representative only of each kind of triangle - the one in which the side a is not parallel to one of the page sides. It was found that the two versions show the same CCP:



indicating that the concept definition has no influence on the hierarchical structure.

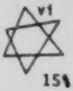
Analysis of the CCP for the whole population (coefficient of reproducibility = 0.9338) shows, that the 19% of students in the CCP do not construct an altitude at all, but something else in all cases, and are therefore uninteresting. The remaining 51% can be subdivided as follows: 16% are able to construct the altitude in each of the 4 triangles, 3% in the first three only, 7% in the first two only and 25% in the isosceles case only.
Table 2 give an analysis of the student responses to this item.

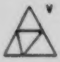
Table 2

				
no construction	5.5%	8%	7%	11.5%
altitude to a	74%	40%	32%	30.5%
median to a	-	20%	21%	20%
perpendicular bisector to a	-	7%	7%	9%
altitude but not to a	-	1.5%	6%	4%
the side a itself	3%	2.5%	3%	3%
others	17.5%	21%	24%	22%

3.3. VERBAL DEFINITION AS A TOOL OF CREATING A NEW CONCEPT IMAGE (research aim d)
The first example requires the student to identify a new concept given its verbal definition.

A bitrian is a geometric shape consisting of two triangles having the same vertex (one point serves as a vertex to both triangles). Identify all the bitrians among the following shapes:

vi

15%

v

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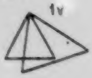
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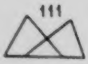
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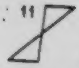
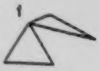

89%

i

82%

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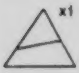
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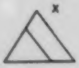
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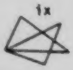
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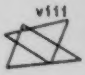
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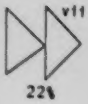

22%

N = 95. 4% did not respond

The percentage of students identifying a shape as a bitrian is indicated in the diagram for each shape. We can see that for most shapes students identify bitrians and non-bitrians correctly.
In shapes (x) and (xi) the distraction property appears again. In these cases

one triangle is part of the other and this inhibits identification.

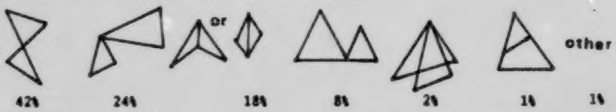
Guttman scale analysis shows a high level of validity (coeff. of rep = 0.9341) for the following CCP: ii ⟶ i ⟶ iv ⟶ ix ⟶ x ⟶ xi

Of 91 students who responded to this item, the concept image of 20% includes the whole cognitive path, 5% identify all shapes excluding xi, 43% have only the 4 easiest shapes, 12% identify shapes ii, i and iv only, 16% only ii and i, and 4% ii only.

We can say that, at least, as far as this example is concerned, the verbal definition of a new concept is a powerful tool in the instantaneous creation of the concept image. Another interesting fact is that we found a CCP structure, not only for concepts whose images have been developed through a period of years, but also for concepts which have just been created.

In version II of the questionnaire the students were required to construct 2 examples of the concept given its verbal definition. Of the 94 students to whom this item was administered, 19% did not respond (only 4% did not respond to the parallel item in version I, indicating that students are more reluctant to commit themselves on construction than on identification tasks). 70% drew at least one correct shape, 59% drew two correct shapes, of which almost half (29%) gave two non-similar shapes, although there was no explicit request so to do. In the questionnaire there was a further "nonsense" example and the results were similar. This would seem to indicate that the verbal definition of a new concept followed by construction task, can be used effectively in creating a concept image.

144 shapes of a bitrian were constructed, of which 22 were incorrect. The remaining 122 can be classified as follows:



The most popular shapes in the construction task are those which were found to be the "easiest" in the identification task. This would seem to indicate that the same cognitive structure applies to both types of tasks.

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CONFLICTS BETWEEN DEFINITIONS AND INTUITIONS — THE CASE OF THE TANGENT

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§ 1. Introduction

One of the important roles of definitions in Mathematics is to be an ultimate criterion to determine whether something is a specific example of a given concept. For instance, a set B is defined to be a subset of A if and only if every member of B is also a member of A. According to this definition every set is a subset of itself. However many students do not see it this way. They do not consider a set as a subset of itself. Another example: a set A in the Euclidean plane is defined to be a convex set if and only if whenever b and c are points in A then the entire line segment between b and c is a subset of A. According to this definition any line segment is a convex set. However, again, this is not the way many students conceive it.

For many Mathematics teachers (in high school and college) a failure to use definitions as an ultimate criterion in cases such as the above is a serious failure. It indicates that the student is unaware of a very important aspect of the methematical discipline. This is probably true but it has been claimed (Vinner 1976 , Vinner 1980, Vinner and Hershkowitz 1980) that there is no psychological ground for the expectation that high school students or college students who are not Math. majors will conceive mathematical definitions the same way the professional mathematicians do it. It was claimed that in the psychological process of concept formation the definition has a role very similar to the role a scaffolding has in the construction of a building. The moment the building is finished the scaffolding is removed. Hence, examples of the concept are intuitively identified or constructed by the student. In terms of Fischbein theory of intuition (Fischbein 1978) our claim is that quite often the response to a mathematical task is immediate and global (not analytic). Fischbein distinguishes between primary intuitions and secondary intuitions. "Primary intuitions refer to those cognitive beliefs which develop themselves in human beings in a natural way before and independently of systematic instruction. . . . Secondary intuitions are those which are developed as a result of systematic intellectual training" (Fischbein 1978, pp. 161, 162). Our point is that quite often in the course of intellectual training the expected secondary intuitions (as an immediate global knowledge) are not formed at all or at least they do not stabilize. This paper comes to illustrate this point by means of one example, the example of the tangent.

§ 2. The Case of the Tangent

Many students learn about the tangent to a circle at a relatively early stage of their mathematical studies. This experience is probably the basis for the intuition of the tangent to additional curves. If a student, who learned only about the tangent to a circle, is asked to draw a tangent to an ellipse or to a parabola he will hardly find any difficulty doing it. However, such a student has to wait for a calculus course until he faces the general notion of the tangent. At this stage there are at least two ways of introducing it. The first one is to assume that the notion is intuitively known to the student. In this case the teacher's role is only to develop a method for calculating the slope of the tangent, namely, to tie it to the notion of the derivative (see for instance M. R. Spiegel, 1963, p. 68). The second way of introducing the tangent is to define it (geometrically) as the limit of secants and only later on to tie it with the notion of the derivative (see for instance A. Schwartz, 1960, p. 7). The question is whether these definitions contribute anything to the formation of secondary intuitions about the tangent. Do they change the primary intuitions based on the tangent to the circle? Can they be applied in complicated cases like the tangent tothe graph of $y = x^3$ at (0,0) (which is the x-axis), the tangent to the graph of $y = \sqrt{|x|}$ at (0,0) (which is the y-axis) or the tangent to the graph of $y = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ at (0,0) (which is, again, the x-axis)? The answer to this question is given in § 4.

§ 3. The Questionnaire and the Sample

The following questionnaire was administered to 278 first year college students in calculus courses which were service courses for Chemistry, Biology, Earth Sciences and Statistics (it was administered after they learned about derivatives and tangents):

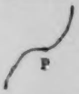
Here are three curves. On each one of them a point P is denoted. Next to each one of them there are three statements. Circle the statement which seems true to you and follow the instruction in the parentheses.

A. Through P it is possible to draw exactly one tangent to the curve (draw it).

B. Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinity. Draw all of them in case their number is finite and some of them in case it is infinite).

C. It is impossible to draw through P a tangent to the curve.

1.



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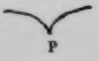
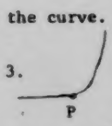


3.



4. What is the definition of the tangent as you remember it from this course or from previous courses.

5. If you do not remember the definition of the tangent try to define it yourself.

§ The Results

Table 1 - Distribution of student answers to questions 1-3

(N = 278)

Question 1			Question 2			Question 3		
A	B	C	A	B	C	A	B	C
65%	10%	21%	13%	52%	29%	48%	26%	15%

Note that the numbers do not add up to 100% since some students did not answer some questions. In addition to that of course, the choice A did not guarantee a correct answer.

Table 2 - Distribution of student drawings to question 1
(N = 278)

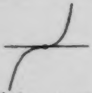
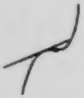
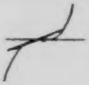
 The right answer		 2 tangents	Another drawing	No drawing
18%	38%	6%	10%	28%

Table 3 - Distribution of student drawings to question 2
(N = 278)

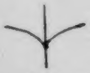
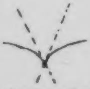
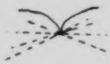
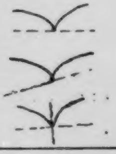
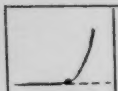
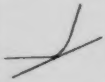
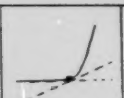
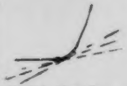
 The right answer	 2 tangents	 Infinitely many tangents		No drawing
8%	18%	18%	14%	42%

Table 4 - Distribution of student drawings to question 3
(N = 278)

 The right answer		 2 tangents	 Infinitely many tangents	Another drawing	No drawing
12%	33%	16%	7%	4%	27%

The student definitions given as answers to questions 4 and 5 were classified into 2 main categories:

Category I: A geometrical definition which expresses a global intuition based on the intuition of the tangent to the circle (touching but not intersection, meeting the curve but not cutting it, being on one side of the curve, etc.).

Category II: Any version of the course definition (mainly the 2 approaches mentioned in § 2).

Table 5 - Distribution of student definitions
(N = 278)

Category I	Category II	Another definition	No definition
35%	41%	19%	4%

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THE USE OF THE MICROCOMPUTER IN COMPUTER-ASSISTED MATHEMATICS INSTRUCTION

Carolyn L. Pinchback
Brooklyn College of the City University of New York

The talk will focus on the research for investigating the effect of the computer as an enhancement of student achievement through the use of the computer as an adjunct to the regular mathematics program for elementary students (grades 4, 5, and 6).

In presenting the results of the study at the conference, two questions will be discussed in detail. One question is whether computer-assisted instruction improves achievement of mathematical concepts. The second question is whether there is an effect on attitudes towards mathematics with the use of the computer.

GEOMETRIC ACTIVITIES IN PRIMARY SCHOOL

G. Plancke-Schuyten, State University Gent
I. Vossen, State University Gent

The paper reports on the PELAG-project 'geometry at the early grades in primary school' of the State University of Gent. PELAG stands for Pedagogical Laboratory of State University Gent and is attached to the Department of Education and Psychology.
Essential for this laboratory is the presence during the day of ± 50 children, which are meant as study objects for the students and the university staff in the field of tests in education and psychology, curricula, teaching-learning processes ...
Observation and experimentation is indispensable for the students during their formation educationists or psychologues and for the university staff in their research work.
As these ± 50 children are during the day at our disposal, care is taken for their education. The children are grouped according to their age: group A age 6-8, group B age 8-10, group C age 10-12.
To each group several collaborators are attached. The University staff of PELAG is characterized by its interdisciplinarity. There is a variety in the academic background of the collaborators: education, psychology, languages, mathematics, biology and physical education.
The team of the geometry project consists of two educationists and two mathematicians.

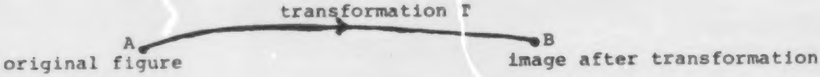
During the period 1965-1973 we devoted much time in reshaping our mathematics curriculum. As we were unhappy with the place of geometry in this curriculum, we intended to develop a geometry program at primary school which should be mathematically cohesive, in which geometrical concepts are in focus, which should relate geometric ideas to the real world and last but not least which should be fitted to the thinking and learning of the primary school-child.
We started our geometry program in September 1978 in the first grade and went on with the same children in September 1979 in the second grade. Due to organisational internal difficulties the program for

the third grade is experimented in the period September 1981 - July 1982.
Some general ideas of the program will be outlined. Reports on the theoretical framework and on the activities executed in the first and second grade are available. These more comprehensive reports may be ordered at the address: H. Dunantlaan 1, 9000 Gent, Belgium.

Mathematical background

The statement "Geometry is the study of properties which are invariant under a certain group of transformations" (Klein 1872) is used as an organizing principle for the geometry program.
A geometry program for primary school can be based on the transformation concept and thus stresses the mutual relations of geometries. Three kinds of geometries of the plane are dealt with: metric geometry, affine geometry and topology.
The purpose is that children can organize subsets of the plane by means of geometrical concepts and that children know what happens to the properties of these subsets under certain transformations.

Children become familiar with three kinds of geometries: motion geometry, sun geometry, elastic geometry. In the two first grades (ages 6-8) motion and elastic geometry are dealt with.
The option to start with reflection in the first grade was inspired by the very inviting character for the children for this kind of transformations. After the phenomenon 'reflection' is being made familiar to the children, the invariance of certain properties under reflection are dealt with. In the beginning the product of the reflection is into focus. The reflection as a process comes slowly to the front during the two first grades; attention is paid to the distance of a point to the axis and to the construction of the image. The construction itself is not an objective at the end of the second grade (age 8). As to the transformation in elastic geometry the children experience them as shrinkings and stretchings without tearing which can result in shape distortions.
In the second grade the following relation is dealt with



Children are asked to find the third element when two of the three elements A, T, B are given. They start with activities where A and T are given and they have to predict the kind of figures they can get. More difficult activities follow, such as: can you get (B) from (A). First they can find it out with concrete material, afterwards they have to do it mentally.

The second grade ends with the concept 'Parallelogram'. The kernels by which this concept is built up are: 'boundary', 'line segment' and 'parallelism'. Terms such as 'vertex', 'open' and 'closed line' are not used. The intuitive notion of boundary is seen in the first two years as a line, where you are trapped, which divides the plane in an inside region where you can walk around and an outside region where you can't get. From the third year onwards the children study the boundary as a line which divides the plane into two or more regions; they also have experiences with straight lines and angles as boundaries.
Starting from the polygon as a boundary, consisting of line segments which fit together, the quadrilaterals are introduced. After the introduction of parallel lines the subset of quadrilaterals with at least one pair parallel lines are dealt with, i.e. trapezoïd followed by the subset of quadrilaterals with two pairs of parallel lines, i.e. parallelogram.

Pedagogical and psychological background.

The two statements "The same subject-matter can be 'known' at different levels" (Varga 1971, 22) and "Emotion constitutes the 'energetics' of thought while cognition provides the structure" are used as psycho-pedagogical starting points (Cowan 1978, p.52).

In building up a program we accepted the following point of view of Shulman (1970, 42): 'to determine whether a child is ready to learn a particular concept of principle, one analyses the structure of that to be taught and compares it with what is already known about the cognitive structure of the child of that age' (see also Papert 1980). Evidence in using the above mentioned mathematical structure (cfr. mathematical background) is found by Martin (1976,108-110). Both structures, the mathematical and cognitive, stress invariance under transformations.

Emotional involvement is important for the cognitive and affective development of the primary child. These aspects of the geometry program are more comprehensively treated elsewhere (Vossen). Movement , gross motor activities followed by fine motor activities, perceptual exploration by means of psychomotor, visual and tactile exploration of the subject matter, stories in the world of the animals, the world of gnomes, the circus, space, adventures of peers, continuous appearance and manipulation of two familiar personages (little John and King Sunnynoddle), games (playing with marbles, rope-dancing, tugging at a rope, playing with hoops, breaking spaghetti-pipes, blindfold touching, playing monkey, blowing up balloons, stretching sheets of rubber, pushing in foamy rubber, ...) are the vital part of the program.

We inserted constantly movement-activities, not only to facilitate concept formation, but also because we are convinced that movement has an over-all importance, in all learning domains, allied to or apart from the cognitive domain. Movement meets a natural need of the child, not only in giving opportunities to explore the world around, but also because the child simply takes pleasure in it, at that very moment. Many children have such a primary need for activity, that they cannot bring themselves to the necessary mental exertion when put in a passive learning situation.

In the presentation and the set-up of our geometric activities we realized an approach which is child-centered and mathematically cohesive. We succeeded not only in an enthusiastic co-operation of the children and a strong affinity towards geometry, but also in a fluent acquisition of the objectives.

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FORMATION OF THE "VARIABLE" CONCEPT USING MATHEMATICAL GAMES

Naomi Taizi & Maxim Bruckheimer
Weizmann Institute, Rehovot, Israel

Introduction

One of the problems in elementary algebra is the understanding of the "variable" concept and its role in mathematical models such as open phrases, functions etc. The problem is particularly serious among socially-deprived students, who have difficulty in grasping abstract explanations.

Most students, including socially-deprived, can correctly substitute a single number in a (simple) open phrase and find the image under a (simple) function. But, when the situation is slightly more complicated - for instance, given a set of numbers instead of just one, or when the inverse is required - the difficulties begin. They are noticeable even when we change instructions, such as when we ask for oral calculations instead of pencil and paper.

A fair amount of experience has accumulated in the use of games in school to overcome such problems. The main reason that makes the game useful in this context, is its potential to create an "external, concrete" situation. To take a simple example, given the open phrase 30-x, will substitution of a positive or a negative number always give a positive result? Instead of the abstract exercise, the student is given the same problem with two stacks of cards face down in front of him. He knows that one consists of positive and the other of negative numbers. In the first place, the very fact that the cards are there physically in front of him gives him a certain security. Also having guessed the answer, the fact that he can turn over and check his guess by substituting the number on the card, provides him with reinforcement. To understand better the contribution of the games we shall give a short description of five of them, all dealing with *open phrases (or functions).*

Five games

1. The Steeplechase (2-4 players)

   This is among the most successful games we have developed (Friedlander, 1977). It consists of a board (Fig. 1), 4 runners and cards on which positive and negative numbers and zero are printed.

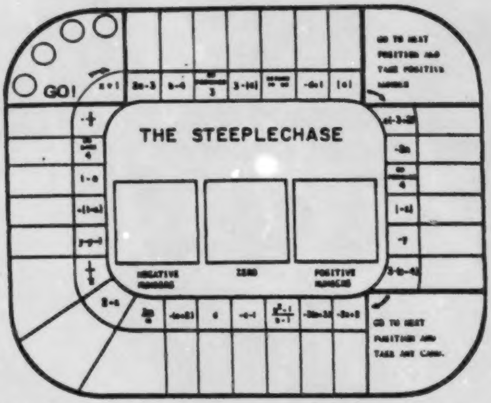


Fig. 1

The cards are placed face down in the appropriate places. Each player in his turn chooses a card from the pile which he thinks advantageous, according to the position of his runner on the board. He substitutes the number in the open phrase on which he stands and the result dictates his move: if positive he moves forward an appropriate number of places, if negative backward, and does not move if zero.

The winner is the one who first completes two circuits of the board.

This game can be played in several variations according to the level of the students (Ilani et al, 1982).

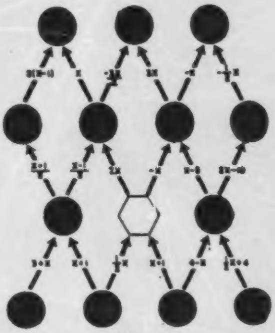


Fig. 2

2. Chip and Arrow (2 players)

This game deals with the relation between a given number and the outcome of its substitution in an open phrase. The game contains a board (Fig. 2), 4 hexagonal chips with numbers (2, 3, -2, 1) which were used at the beginning of the game, and circular chips with various (well-chosen) numbers. The first player places one of the hexagonal chips on the hexagon on the board. Initially, each player receives 5 chips and moves by putting the appropriate number at the head or tail of an arrow, at the other end of which a number has already been placed. If a player does not have an appropriate chip, he takes from the bank until it is exhausted. The winner is the one who first places all his chips.

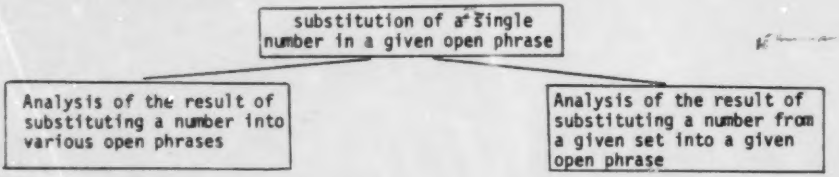
3. Bigger is Better: Mark I - (2 players)
   Each player is given a pile of cards with open phrases face down. There is another pile of cards with numbers, also face down, in the middle. At each turn, both players turn over one of their own cards and one of the number cards in the middle, each substitutes the number in his open phrase, and the one who gets the bigger result take the cards. The winner is the one who has the most cards at the end.

4. Bigger is Better: Mark II - (2 players)
   The game consists of 16 cards with open phrases and a die with numbers from -3 to +3.
   Each player takes 2 cards. The die is thrown and the players have to consider into which of their two open phrases to substitute to obtain the bigger result The player who has the bigger number takes the cards. The game continues until the cards are exhausted. The winner is the one with the most cards at the end of the game.

5. Bigger is Better: Mark III - (2 players)
   Here the role of the numbers and open phrases is reversed. The players hold numbers and have to choose which to substitute in a given open phrase in order to get the most advantageous result.

Discussion

The various aspects of the variable concept with which the games deal are illustrated in the following diagram.


substitution of a single number in a given open phrase
Analysis of the result of substituting a number into various open phrases
Analysis of the result of substituting a number from a given set into a given open phrase

Further, the games also deal with the converse problem, estimating the "source", given the result of the substitution, for the cases mentioned in the diagram.

All the games deal with the simple act of substitution of a given number in a given open phrase. Nevertheless, even in this simple aspect, the games are useful for weak students. If a number is written on a card the student can touch it and move it about. For example, in the game *Bigger is Better: Mark I,* we noticed that weak students have a tendency to put the dice physically on the card. This seems to provide the students with an intermediary stage between pen and paper exercises and oral exercises, and thus they gain confidence and "get a feeling" for the meaning of algebraic expressions.

In the *Steeplechase* and *Bigger is Better: Mark III* we have a situation in which the player has to choose from *a set of numbers*. In the latter the student has 5 cards, in other words a limited choice. It was interesting to observe that students were substituting less and less as the game progressed. First, they eliminated a number of possibilities by estimation or prediction. In the case of the *Steeplechase*, however, the player does not have any numbers in front of him, but has to choose a number from one of three piles, whose contents he knows but cannot see. In this case students who have difficulties, came to the conclusion by trial and error. Thus, after a little while they began to assume certain mathematical rules of their own, which in some cases were wrong. One such frequently recurring rule was, if a negative sign appears in the expression one must substitute a negative number in order to get a positive result. Therefore, in the case of $z - 4$ one must substitute a negative number!

In some cases we noticed a progressive development of these rules, wrong ones being corrected and others being modified in the light of experience.

In almost all observed cases the students learned from experience which was the "correct" pile from which to choose.

An interesting case was a student who deliberately picked a number from the "wrong" pile and said that he knew it was wrong, but he did not know how to handle negative numbers.

This latter example shows that the student did understand that he should choose a negative number, but lacked the technical skill, or confidence, to work with them.

The feeling of what an open phrase does to numbers is created in the *Steeplechase, Bigger is Better: Marks II and III,* and *Chip and Arrow*. In these games the students learn, while playing, the significance of "-" in front of the variable, or the influence of a number between zero and one as the coefficient of the variable. Again, in *Bigger is Better: Mark II* the student has 2 open phrases and if he cannot decide which of the two to use he can substitute in both, whereas in the other games he has only one open phrase in each case but he needs to make a decision which number of a set of numbers to use. This encourages the student to deduce "rules" of his own and to ask himself what is the effect of the open phrase.

Remarks such as "in b^2 it is better to substitute -3 than 1, in order to get a larger number" and similar remarks are evidence of decision making. Many such remarks were overheard. Another remark concerns the expression $3 - b^2$, where the student was interested in a positive result. "A pity I haven't a zero!" (What is interesting about this remark is that he did not say - a pity I have no negative numbers - as one might have predicted would happen.)

These sort of remarks, which changed and "corrected" themselves from stage to stage, are evidence of the fact that the combination of the possibility to make immediate checks and to "handle" the numbers, or, to relate to a given pile of numbers, leads the students from the level of pure technique to a higher stage of analysis.

The "reverse" direction is mainly treated in *Chip and Arrow*. Most of the time the game can be played in the direction of substitute ⟶ result. But sooner or later he reaches a more difficult stage, when the only unoccupied spaces are at the heads of arrows. In which case, he has to pick a number such that when substituted into the open phrase he will get the given number. Here we have a classical classroom difficulty in which the student muddles up the outcome with the number to be substituted. And this also occurred initially in the game. Even though the number was at the head of the arrow, all the students did was to substitute in the expression and place a counter correspondingly at the tail of the arrow.

Here we have another of the advantages of using games. The student who makes this mistake is almost inevitably corrected by his opponent. The latter has only to check (and usually does so) whether the number just placed, when substituted in the appropriate expression, gives the correct answer. This is, of course, a much easier task than that of the player whose turn it was. This gave us a situation in which either the students could correct themselves, as happened with the more able students, or the teacher could effectively intervene.

Essentially, the student faced with this situation in the game, has three possibilities. He can guess what number to substitute, check his guess and then see whether he has that number. He can use pencil and paper, write down the appropriate equation, calculate and find the number to substitute. And the simplest of all the possibilities to substitute each of the numbers in his possession and check whether one of them gives the desired result. Gradually, the players develop one of these techniques.

Conclusion

The results described were mainly obtained by observation and interviews with students across a wide ability spectrum, ranging from relatively very able (students) to extremely weak, and taking in socially deprived students. In all these groups we found that with an appropriate mix of games one could achieve at least partially, the following:

i - A pleasant way of practising basic algebraic skills, leading to mastery.

ii - A feel for simple algebraic expressions in the sense that, a student is aware of possible outcomes of various substitutions and the symbols thus having a meaning for him.

iii - The strategies which the games encourage in order to be played successfully, also encourage the student to go beyond the level of pure technical skills to the level of analysis.

Reference

Friedlander, A. "The Steeplechase", *Mathematics Teaching* (September 1977): 37-39

Ilani, B., Taizi, N. and Bruckheimer, M. "Variations of a Game as a Strategy for Teaching Skills". In *Mathematics for the Middle Grades* (NCTM 1982 Yearbook): pp. 220-225.

B. PROBLEM SOLVING

A cross-cultural study of problem solving involving pupils aged 11-13 years

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David PIMM, Nottingham University, U.K.

Increasing interest is being shown by teachers in using problem solving at all levels as a vehicle for teaching mathematics. This is publicly supported in documents such as the recent Cockcroft Inquiry Report into the teaching of mathematics in schools in England and Wales (H.M.S.O., 1982), which described problem solving as one dimension missing from many classrooms. However, little agreement yet exists as to the benefits to pupils or, indeed, what part problem solving can play in mathematics teaching/learning. Indeed, major differences exist regarding what problem solving is; for example, between

word problems (Travers (ed), 2nd Handbook of Research on Teaching, Chaps. 35 and 36, Rand, McNally, 1973),

investigations (Biggs, E., Investigation and problem solving in mathematical education in Howson (ed), Mathematical Education, Proceedings of ICME II, C.U.P., 1973),

'real' problem solving (Burkhardt, H., The Real World and Mathematics, Uni. of Nottingham, 1979),

mathematical thinking (Mason J., Burton L., and Stacey K., Thinking Mathematically, Addison Wesley, 1982).

For a full coverage of the many interpretations see Hill, C.C., Problem Solving: Learning and Teaching, Frances Pinter, 1979. Teachers often claim that problem solving is suitable only for the brightest pupils or that there is too little time available to cover the conventional syllabus without including 'unconventional' materials.

In one of the Begle Memorial lectures at ICME V, Stephen Brown succintly characterised the mainstream of research into problem solving by the following diagram :

$$\begin{tikzcd}
\text{Problem} \arrow[r] & \text{Solver} \arrow[dl] \\
\text{Solution} &
\end{tikzcd}$$

in other words, presenting someone with a problem and watching their attempts to solve it. Although he rightly pointed out that this is by no means the only method of observing problem solving, it is the one which we have chosen for this study. Our interest lay in moving closer to activities which are more akin to mathematical behaviour than are observed in most classrooms thus demonstrating to teachers what can be achieved by different means in a format which they could use. The difficulties were underlined by David Wheeler in a presentation at PME V (Hillel, J., Processes involved in the solution of non-routine problems). We acknowledge the constraints of Brown and Wheeler/Hillel, but were nonetheless interested in constructing a study on a very small scale which might expose differences and similarities in :

(a) different cultures

and

(b) different investigational methods.

We were also concerned that any insights that might be achieved should be of value to teachers.

The study has been implemented by five colleagues in four countries (see above). The following five problems were posed (each researcher was responsible for the particular formulation (s)he employed) :

1. If you know how many sides a particular polygon has, can you predict how many diagonals it could have?
2. Fill in the spaces in this 5 figure number

   45..8

   so that 9 divides into it exactly. In how many different ways can you do this?
3. A class of children were working on the area and circumference of rectangles.

   Six children announced the following :

DAVID : Two rectangles with the same circumference have the same area.

SUSAN : Two rectangles with the same area have the same circumference.

GUY : Enlarging the circumference of a rectangle always makes the area increase also.

SERGE : Enlarging the area of a rectangle always makes the circumference larger.

BRIGID: Every rectangle with an area of 36 cm^2 has a circumference of not less than 24 cm.

LOUISE: For any rectangle there is another one of equal area but with a larger circumference.

Do you agree or disagree with each of these children? Explain why.

4. There is room in a grill to toast one side of 2 slices of bread in 2 minutes.
   How long does it take to toast both sides of 3 slices of bread? Is that the quickest way? Work out the quickest way to toast 4 slices of bread on both sides. Then try 5 slices of bread and then 6. Can you predict the quickest time for any number of slices?

5. In a village there are 3 streets. All the streets are straight. One lamp post is put at each crossroad. What is the greatest number of lamp posts that could be needed? Now try 4 streets. What about 5 streets. Predict the answer for 6 streets and then check it. Can you see a pattern? Why does the pattern work?

Three hypotheses were investigated :

1. In all four countries, pupils will perform similarly, age for age, at investigations of this kind.
2. Style of working with pupils makes a difference to their performance.
3. Pupils respond favourably to problem solving of this kind.

Each investigator chose a method for presenting and exploring the problems with the pupils. In order to provide a baseline of comparison in each country, a group of pupils in a mixed ability class were presented with the five problems in a conventional paper and pencil format. These children were also asked to write a paragraph responding to the question :

"Is there a difference between doing problems like these and what you usually do in mathematics class?"

Where possible, each investigator asked of subjects in the study :

"Did you enjoy doing the problem? Give your reasons,"

in order to explore how the pupils responded to such an approach. Styles of presentation varied from individual interviews, pairs (Balacheff, PME, 1981), to whole class methods. Examples of some responses obtained in England are given below. First, transcript and written work of two pupils aged 12 years, who were interviewed individually, are presented. After these are 2 protocols obtained from class-based lessons. Problem 1 was attempted by two 12 year old girls and Problem 4 by two 14 year old boys.

I. Make sure it is clear and then have a go at it.
...
S. There's three for the greatest number
I. Is there? Uh, huh.
S. I think and one for the smallest number.
I. Why do you think there's three?
S. Because if you had one going like that ...
I. Could you draw a picture?
S. If you had had them going like that, one there, one there and one there ..
I. Uh, huh - one at each corner.
S. And if you had ... If just one, it'd go like that, oh, hang on a minute, it'll still go like that.
I. So all three roads come into one?
S. Yes.
I. So could you put 3 down for that and one for the smallest? How would you convince somebody who wasn't sure that 3 was the most?
S. Ur, I don't know. Well, it's obvious because it's 3 roads and if you had them all crossing over then you'd need 3.
I. O.K. fine. What about 4 roads?
S. Well, one would still be the smallest, wouldn't it?
I. Uh, huh.
S. Shall I draw that? (c) I'm not quite sure about this. I think about 5 because you'd have a main road going across there then you'd have to have an extra roundabout just there. So that I draw that?

I. Do you think you'd ever need 6?

S. What, six roundabouts? Yeah, probably for five roads.

I. But how about 4 roads?

S. I don't think so.

I. Uh, huh. How would you convince ...

S. Well, I'd just tell them that that's what it was.

I. And what if they said they thought you might need six?

S. I'd just say that the easiest way of doing it would be so that you'd only need 5.

I. O.K. How about 5 roads?

S. One would still be the most you'd need, I mean, the least you'd need. It's 7 on there, there's one going down there - or I suppose you could have it going down there and then you'd need 3 or 4.

I. Could you draw a picture?

S. Like that.

I. How many have you got?

S. 9.

I. 9.

S. Do you want me to draw one?

I. Why not? And again, 9 is the most you could get?

S. No, you could probably get some more but it wouldn't be very good planning of the roads.

I. Suppose you had to budget to cover all possibilities and you wanted to know the most you might have to spend?

S. Um, nine's the most.

I. If you had to convince your other committee members ...

S. Oh, I'd just say, well that nine is probably the most you'd need and show them that! That's how they're going to have the roads.

I. Fine, what do you think it would be for six roads?

S. Um, have another one going down there. shall I draw that?

I. How many do you think that it would be?

S. Well, another 4 roundabouts.

I. Could you draw that?

S. Wait a minute ...

I. How many have you got there?

S. 12.

I. You thought you'd have 4 more, 13?

S. Mm, well I missed one out just there.

I. So that's thirteen. Can you see some pattern in the numbers?

S. Um? Yeah. It goes up equally 2, 4, 6.

I. Sorry, which goes up?

S. Well, if you have 3 to 5 that's 2 and that's 4 and that's 6. Oh no, that isn't 6 is it. They're all multiples of 2.

I. Uh, huh. What's all multiples of 2?

S. All these numbers of roundabouts.

I. The actual numbers you're getting?

S. Yeah - no, not those, the differences.

I. I see. So if I wanted to know how many for 7 roads, could you tell me?

S. It might be 4 again.

I. Could it be anything else?

S. It might be 6.

I. How could you find out?

S. Do it for 7.

I. Which do you think it's going to be?

S. Probably 6.

I. Why 6?

S. Because 7 is more than the rest and it'll probably be more than 4.

I. I see. Have a go at the picture.

S. 17 I think.

I. It's difficult to count - could you go through it again?

S. No, 18. Shall I put that down?

I. Does it fit the pattern?

S. No, it's gone up by 5.

I. So what do you think is happening?

S. Um ... well, those 3 numbers are all 3 times table.

I. Uh, huh.

S. Well, 3 + 5 = 8 and ... I can't see any pattern other than that. There may not be an easy pattern between them. they just get bigger.

I. If you were going to put one more line in to do it for 8 roads, where would you put it?

S. Um.

I. Just do it in red with a dotted line where you'd put your extra line. Does it make any difference where you put the line?

S. Yes, because if you put it down there it wouldn't be as many.

I. Uh, huh. Why not?

S. Becasue it doesn't cross over as many of the roads.

I. I see.

S. If you put it there, there's quite a few crossroads on there as well.

I. Where would you put it so it crossed the most roads?

S. Well down there because ... There's 5 crossroads just in there.

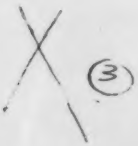
NAME: Nancy HAWKES:

PROBLEM:

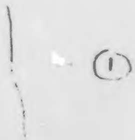
In a town there are three main roads. All the roads are straight. A roundabout is to be built at each crossroads. What is the greatest number of roundabouts which could be needed? What is the smallest number?

Now try four roads. What about five roads? Predict the answer for six roads, then check it. Can you see a pattern? Why does the pattern work?

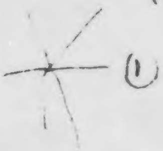
(a)

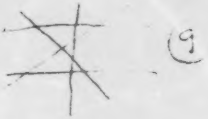


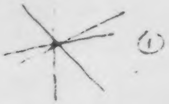
(b)

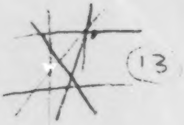


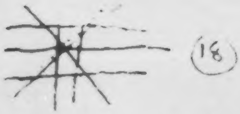
(c)











S. Does it mean that all the roads go into one, or they're separate?

I. Which way would give you the greatest number of roundabouts possible?

S. Well, I think there'd be quite a few for each of the main roads because there'd be roads going off.

I. I see; if you'd just concentrateon the main roads, just worry about where the main roads cross, then how many will there be?

S. (immediately) One.

I. Uh, huh. Can you draw a picture for that?

S. Could be two though for the other one.

I. Uh, huh. O.K. So how many would you need altogether? If you had three main roads? What's the most you could possibly need?

S. Two.

I. 2, uh, huh. How would that look?

S. Like that, that's all one road.

I. How would you cinvince someone that two was the most you needed?

S. Well, you wouldn't need 3 because that would mean there were 4 roads. If there's 3 roundabouts that's 4 roads.

I. Uh huh. Would there have to be?

S. Mm ... yes ... so is that the pattern, if you had four roads, you'd need three roundabouts?

I. Yes, what they want you to look for is a pattern between ...

S. Yes.

I. What would be the smallest number of roundabouts you'd need with 3 roads?

S. 2.

I. So that's the most and the smallest?

S. Yes, if you want to involve all the roads.

I. Uh, huh.

S. Otherwise one would just go straight down without going near the other two.

I. A sort of fly-over you mean?

...

S. Well, I think it's two.

I. Fine O.K. Can you write down 2 so there's a record of it? And that's the most and the least?

S. Yes.

I. Fine, can you now move on to 4 roads?

S. Do you want me to draw it?

I. Yeah.

...

I. And is 3 the most you'd ever need?

S. Yes.

I. How about the smallest number?

S. 3.

I. 3 again? O.K. Could you write that down. How about 5 roads?

S. That'd be 4.

I. 4 again, there'd have to be 4? Can you see a pattern ...

S. Yes, it's one ... if you have 2 roundabouts that means there'll be 3 roads and if you have 3 roundabouts involving 4 roads and if you have 4 that would be 5.

I. And if you had 17 roads ...

S. That would be 18.

I. So what's the rule that gets you from the number of roads to the number of roundabouts?

S. The number of roads it is you take 1 off, you take the number off, 1,the number (laughs)

I. O.K. Can you write that down as if you were trying to explain it to someone who hadn't thought it out?

S. Take the number of roundabouts and add 1 on to it.

I. And what would that give you?

S. Um, the number of roads.

I. The number of roads, O.K. ... And why do you think that pattern works?

S. Because if you needed the roads to all be connected in that way, um, you'd just need the, you'd take one away from the number of roads (hesitant).

I. Uh, huh, why is it one less rather than two less or one more or the same number of roundabouts as roads?

S. Cos then it would leave one road on its own.

I. Uh huh.

S. Because if there's only one, it can only have two roads.

I. Yes, I see.

S. For it and the other road would just be on its own.

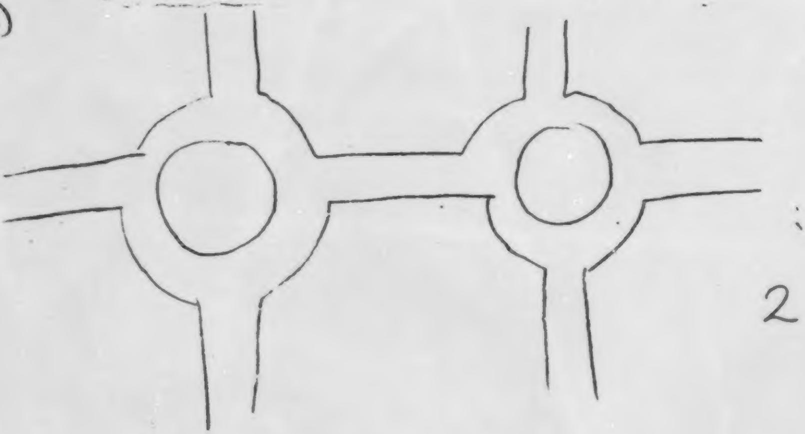
I. Then if you want to link it up ...

S. You need to take one off the number of roads, for the number of roundabouts.

ROUNDABOUTS

Pupil Answer

(a)



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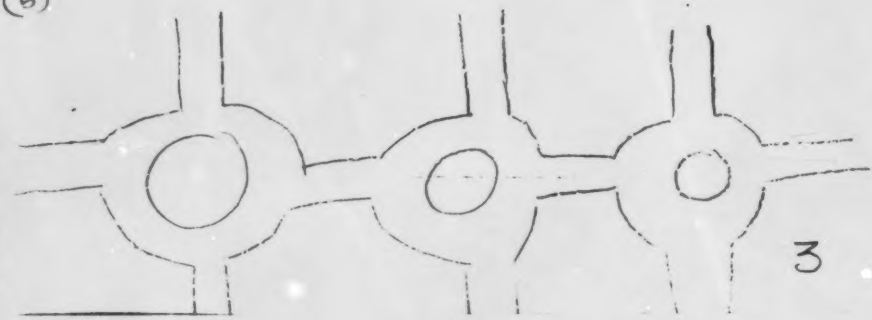


(c)

take the number of rounderbouts and add
(d)
1 on to it and this would give you the D.
number of roads

Problem 1.

first idea

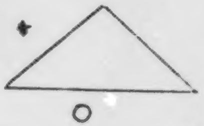
1a look up in the dictionary What a polygon is.

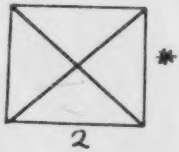
2a. look in a maths book.

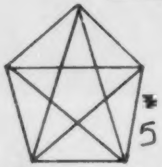
3a look in a Maths dictionary.

This idea (3a) worked and we found out that a polygon is a Simple closed curve With 3 or more Sides.

Eg





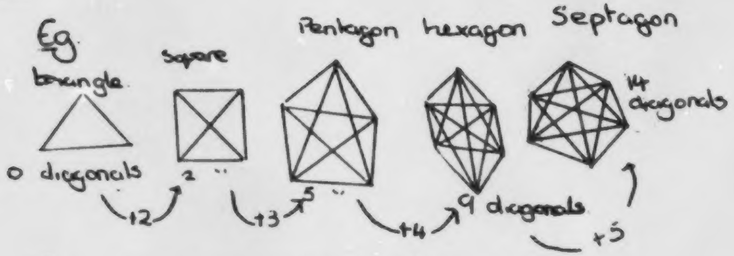


*Then we Predicted how many diagonals a square has what we think is the Answer - 2.

*A triangle has no diagonals.

*A pentagon has 5 diagonals. we noticed a sequence

Problem 1(cont)


Eg.
triangle
square
Pentagon
hexagon
Septagon
0 diagonals
2 "
5 "
9 diagonals
14 diagonals
+2
+3
+4
+5

This sequence carrys on:

+2 +3 +4 +5 +6 +7 +8 etc.

Problem 4

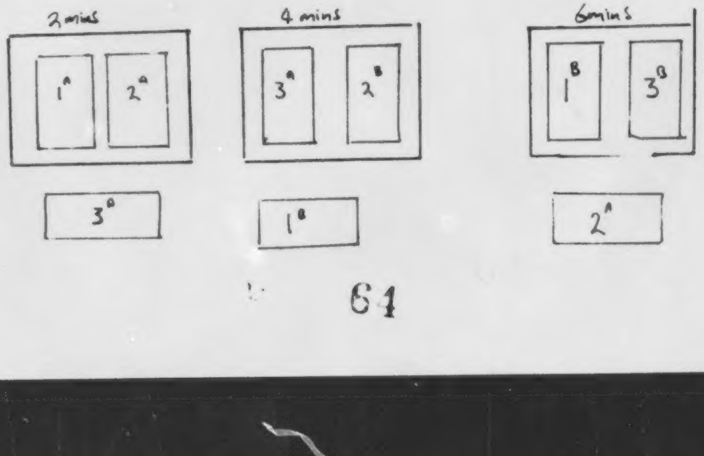
If 2 slices of bread are toasted on one side in two minutes at the same time. It is the equivalent of having one slice done on one side only in two minutes.

It would take four minutes to toast two slices on both sides
" " also take four " " " one " " " "

And so three slices on both sides would take ~~at least~~ eight minutes in ~~the most economic way~~

This is not the most Economic way though.

This is


2 mins
4 mins
6mins
1A
2A
3A
2B
1B
3B
3°
1B
2A

A number of methods for analysing such records will be discussed at the workshop related to this study at PME VI, Antwerp, 1982. Among these are:

1. analysis by problem to see if different problems provoke alternative strategies;
2. analysis by method of presentation to see if different techniques of introducing pupils to such problems affect their protocols;
3. analysis by age of pupils looking in particular at the range within one year group as well as at differences between younger and older pupils;
4. analysis of the mathematical thinking skills exemplified.

In each case, differences between countries will be highlighted and discussed.

Intending participants of this workshop are encouraged to attempt to gather some information of their own relating to these problems and this study, so that a fruitful exchange and discussion can take place.

THE POCKET CALCULATOR IN THE CLASS-ROOM : A PROBLEM SOLVING ACTIVITY

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Université Paris 8 (1)

Using a pocket calculator to evaluate an expression is shown to be a complex problem solving activity when a procedure another than the usual one is to be used. An experiment is described, which shows that even for students this task is a difficult one. It is suggested that planning a calculation to be worked out on a pocket calculator may be an interesting situation for training children in solving problems.

Pocket calculators are now everyday items and are available as pedagogical tools. That they are in common use does not mean that they are easy to handle. When one questions owners of pocket calculators, it is surprising how many of them know very little about what they can do with them. What they know is how to use them in very limited contexts such as adding a series of numbers, finding the product or quotient of two numbers. Often they have not explored other possibilities such as chaining different types of operations or using the memory as a storing or computing device. Most of the time they use them in a very mechanical way and don't wonder about what the machine is really doing when they are pressing keys.

We suggest that one of the reasons for this is that it is really difficult to use a pocket calculator in a nontrivial way and that the difficulty lies in the fact that it is actually a problem solving activity of a non elementary type.

A further suggestion is that training students to use pocket calculators may result not only in more efficient use of these machines but also in improved ability to solve problems in general.

Within the information processing approach (Newell and Simon, 1972) problem solving is viewed as searching for a path from an initial state (the givens of the problem) to a goal state (the situation or results to be attained). Along this path there are intermediate states obtained by applying one of the permitted transformations to the preceeding state.

A problem arises when the subject has no ready made path to reach the goal. In most puzzles this is obtained by imposing constraints on the possible transformations so that the obvious solution is not allowed. For example in the missionnaries and cannibals problems, the obvious way to make the transportation from one bank to the other is not possible due to the limited number of seats in the boat and the relations to be satisfied between the number of missionnaries and cannibals on each bank.

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The same type of problem is obtained when students are required to compute the value of an expression, given some constraints that an expert would adopt spontaneously order to have an optimal solution. We shall take as an example the computation of the variance of a sample according to the formula :

$$v_c = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1}$$

where n is the number of observations in the sample and each x_i is a value in the sample from 1 to n.

An obvious way of evaluating the expression is to compute the sum of the values, find the square, divide by n, save the result on a piece of paper, compute the sum of the squares, subtract from it the preceeding result and divide by n-1.

With a pocket calculator having a memory store (and the instruction "add into memory") and the same logic of priority as in algebra, a procedure exists which satisfies two more constraints : introduce only once each x_i value and do not save on paper results to be reintroduced later.

The crucial step in this procedure is to compute at the same time the sume of the values and the sum of their squares. The way to do this is to sum each value into memory before pressing the $\boxed{x^2}$ key and to have the squares added on the display using the $\boxed{+}$ key.

This problem was proposed to first year students following a course of statistics for psychologists. They had learned just before in the same session how to use a TEXAS TI30 with the aid of a booklet similar to the standard one but somewhat simplified.

At first the problem was given without any explicit constraint regarding the procedure : Ss were told only to find a solution with a minimum number of key presses. Most of the Ss (86 %) found a procedure allowing a correct solution but none used the optimal procedure. They were then told to find another method satisfying the two constraints. This problem proved in fact to be very difficult and only 27 % of the subjects gave the optimal solution. This is an average value : there were actually several experimental groups which differed in the learning conditions and the fact that the calculator was available or not when Ss were planning the sequence. There were differences between groups, which will not be detailed here.

Why is this problem so difficult for students with no training in computer programming ?

Due to the constraints *S*s are not allowed to reproduce the usual procedure. They cannot establish a one to one correspondence between each step in hand computation and each action of pressing a key. They have to devise quite a new procedure taking as a reference not the operations done in hand calculation but the functions available in the machine. In order to solve the problem *S*s have to build a complex representation, in which the main goal is divided into subgoals and each subgoal translated into a sequence of key presses. This implies that functions are considered not separately but in their interactions and are combined with each other, so that the result of a series of commands could be anticipated.

There are three levels of difficulty in building this representation.

At first *S*s must have a correct interpretation of what is done by each key press. This is not easy and we have observed many misinterpretations. This topic was studied in a condition where *S*s were given a series of key presses and asked to tell what would be the content of the display and of the memory after each key press. For instance, after a sequence such as $\boxed{4}$ $\boxed{+}$ $\boxed{3}$ many *S*s think that 7 will appear on the display : 72 % of the *S*s wrongly considered the execution as being immediate at least once in the test. When they are operating on the machine such an error is immediately corrected : they automatically press the $\boxed{=}$ key where they see no change on the display. This knowledge operates on the level of action but is not operating on the level of representation : they forget the execution command when programming a sequence of key presses.

Other misinterpretations are related to memory commands. For instance $\boxed{\text{STO}}$ (which transfers into memory the content of the display after losing the previous memory content), is often interpreted as the transport of an object from one place to another, so that the quantity transferred from the display is thought to be no longer available on the display and to have been added to the content of the memory store. $\boxed{\text{STO}}$ is then interpreted as adding into memory, which is actually the function of $\boxed{\text{SUM}}$; this interpretation is shown by 35 % of the *S*s. Conversely $\boxed{\text{SUM}}$ is frequently seen (about 50 % of the *S*s) as operating in the same way as $\boxed{+}$: for instance following the sequence $\boxed{3}$ $\boxed{\text{SUM}}$ $\boxed{5}$ $\boxed{=}$ the content of memory is thought to be 8.

A second type of difficulty lies in the fact that when planning there action, *S*s think of goals and subgoals to be reached but are not able to deduce immediately how to attain each goal from the knowledge they have about how the machine works. They must infer what to to in order to obtain what they want from what they know about how the machine works.

As a rule knowledge about functions is expressed as implications of the form

(1) if p then q

or (2) if p then q and r

To use this knowledge it is necessary to transform it in the following way

(1') if I want q, I may do p

(2') if I want q and r I may do p, but if I want only q, then I must try something else : either see if there is another way to get q or, if not, try to cancel the effet of r.

As an example, let us take the instructions $\boxed{\text{STO}}$ and $\boxed{\text{EXC}}$

$\boxed{\text{STO}}$ → the content of the display is transferred into memory, the content of memory is lost

$\boxed{\text{EXC}}$ → the content of the display is transferred into memory, the content of memory is transferred to the display

From this knowledge about functions I have to deduce rules to make this knowledge usable :

If I want to transfer a result into memory I may use $\boxed{\text{STO}}$ but only when I no longer need what is at present in memory.

If I want to transfer a result into memory and save what is already in memory, I may use $\boxed{\text{EXC}}$

When the effects of each command are rather complex these inferences are not automatically done and Ss may have difficulties in deducing what means to use in order to attain their goals.

A third type of difficulty is related to the overall planning of the activity.

The different stages of action have to be organized in quite a different way than when computation is done by hand. In order to fulfill the constraint that each value has to be typed only once on the keyboard, the procedure must be planned so that each value is used for computing two quantities at the same time, the sum of values and the sum of there squares. When operating with pencil and paper this is never done, the quantities being evaluated one after the other.

So learning to use a pocket calculator efficiently is quite a difficult problem. The difficulties are usually understimated and only a few studies have emphasized this point (Mayer, Bayman 1981; Young 1981; Friemel, Richard, Silvart, Weil-Barais 1982).

From an instructional point of view this type of task seems quite well adapted to teach children how to draw up a procedure, how to divide a task into goals and change the goals as a function of the constraints of the situation, how to program actions to attain a goal from knowledge about functions, how to detect discrepancies between what is expected und what is really obtained through the sequence of instructions, how to find the cause of these discrepancies, and so on.

This type of task is a complex but miniaturized problem solving activity and as it takes place in a natural setting we may expect more interest and more personal investment than is the case with artificial problems.

The caracteristics of interest are :

- the domain of knowledge which is relevant is narrowly circumscribed
- at each time the student may have a feedback by testing his sequence and this is independent of the teacher's judgment
- the causes of errors are more easilyidentified than in usual problem situations and the student is in a better position to become aware of what has led him to the errors.

All those characteristics are probably more favorable than classical problems for training students to evaluate their procedure and develop the metacognitive aspects of problem solving activity.

Problem solving is being more and more perceived as an activity in which metacognition is concerned primarily (see Piaget 1974, Silver, Branca Adams 1980, Richard 1981). We suggest that tasks such as programming a calculation on a non programmable pocket calculator are well suited for the development of metacognition.

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Recherche d'information et planification dans la résolution de problèmes à l'école élémentaire

Searching information and planning in problem's solving activity

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One can consider that solving a problem consists in passing from an initial state to a final one by completing a sequence of steps. The main aim of this study is to research the organisation of this sequence of steps : study of the forms in which this organisation is developed, the different processes involved in it and the condition for their implementation.

For all given problems, each stage deals with a single operation of the same kind each time; the required knowledge is minimum so that the only difficulty lies in selecting and arranging the operations to be performed.

Therefore it is necessary to define the extent to which the goal to be reached (i.e. that which is defined in the instructions) is taken into account for the selection and the arrangement of theses operations. At this stage the word "planning" will be introduced.

Three criteria have been taken into account :

- the structure of the problem : number of operations required for the solution and the degree of interpendence between these operations.

- the nature of the work required from the subject; in the reference situation, the subject is given all the data which can be used for the solution; in other cases the subject is required to ask for some data.

- the specific nature of these situations.

> *The results obtained show that few subject reveal a behavioral pattern based on planning. In most cases the required goal is taken into account as follows :*
> *the subject begins by looking for an operation which he can carry out whithout questionning the usefulness of the results of this operation. This first choice is based on perceptive exploration. The result obtained enables a new operation to be performed. This operation can either be useful or useless for solving the problem, and so on ...*
> *The subject is then confronted with another problem for which the goal is closer and therefore more attainable because it is whithin the limits of anticipation.*

I. - L'objectif de la recherche

L'objectif général de la recherche est l'étude de l'organisation de la séquence des actions dans un problème : étude des formes que pren cette organisation, des différents processus qui y concourrent et des conditions de leur mise en oeuvre.

Ce type de problème a été abordé notamment par HOC (1977, 1979,1981 à propos de situations de programmation où la planification prend des formes explicites (construction d'organigrammes), où elle se trouve bien circonscrite et est nettement distincte de l'exécution. Dans la résolution de problèmes autres que de programmation, on ne dispose pas de trace spécifique de cette activité, en dehors de ce que l'on observe à travers le comportement d'exécution et les verbalisations spontanées ou sollicitées qui l'accompagnent. De ce fai et en raison des interactions complexes qui interviennent entre cet activité et d'autres processus en oeuvre dans la résolution (analyse du problème, évocation en mémoire de connaissances, de procédures o de souvenirs de problèmes analogues), les modes d'organisation de l'action ont rarement été un thème central des études.

La principale exception concerne certains problèmes de transformation d'états (comme la Tour de Hanoï) où il convient de passer d'un état initial à un état final par une suite de transformations (manipulations), dont chacune est très simple mais dont l'ensemble peut être très complexe et où il est crucial pour aboutir à la solution, de se donner des buts intermédiaires (SIMON 1975, BYRNES et SPITZ 1979, KLAHR 1978, 1981). Une autre contribution récente est celle de l'Ecole de Genève qui manifeste un intérêt spécifique pour ce genre de problème dans le cadre de l'approche procédurale qu'elle développe actuellement.

Nous nous proposons d'aborder cette question dans un cadre plus large. Pour cela nous utilisons à la fois des problèmes de changements d'états dans lesquels les transformations sont des changements physiques consécutifs à des manipulations et des problèmes dans lesquels le passage d'un état à un autre consiste dans la déduction d'une information nouvelle, numérique ou non numérique selon les cas. Ainsi nous pouvons étudier dans quelle mesure les processus d'organisation diffèrent selon le type de transformation en jeu (physique ou informationnelle). Les problèmes choisis ne font pas partie de ceux que les élèves ont l'habitude de résoudre ; néanmoins pour certains, les informations à traiter sont familières aux élèves.

Tous les problèmes posés ont les caractéristiques, d'être solubles en une suite d'étapes ; chaque étape comporte une seule opération, qui est de même nature à chaque étape. Ces choix ont été faits pour une double raison.

La première est qu'il convient de minimiser toutes les sources de difficulté autres que celles qui concernent la séquence des actions proprement dite. Il convient de réduire a minima la difficulté du choix de chaque opération, donc d'éliminer les inférences spécifiques dont pourrait dépendre ce choix et les connaissances spécifiques sur lesquelles ces inférences reposent. De la sorte, les connaissances requises sont minimales, les règles de déduction également, si bien que la seule source de difficulté consiste à sélectionner et ordonner les opérations à faire.

La seconde raison est que cette contrainte permet de construire des problèmes mettant en jeu, les uns des transformations physiques, les autres des transformations informationnelles, mais qui sont en tous points comparables par ailleurs.

2. - La problèmatique

La question centrale est de savoir dans quelle mesure le but à atteindre, celui qui est défini dans la consigne, est pris en compte dans la sélection et l'organisation des opérations effectuées par le sujet. On parlera alors de planification.

Il convient de souligner que la planification est ici d'emblée une planification détaillée. Autrement dit, les opérations qui figurent dans le plan sont des opérations directement exécutables par le sujet qui dispose d'un algorithme pour ce faire. Ce ne sont pas, comme dans la programmation, des opérations macroscopiques telles que celles qui figurent dans l'organigramme, et qu'il faut détailler en une suite d'instructions, qui seront les opérations exécutables. Dans ce cas les opérations macroscopiques sont en fait des sous problèmes dont la solution requiert l'élaboration d'un nouveau plan pour passer au niveau de détail des opérations exécutables. Pour les situations qui nous occupent la planification ne produit pas des plans abrégés au sens que HOC (1981) donne à ce terme mais plutôt des schémas d'éxécution. La planification peut intervenir dès le début avant toute exécution d'opération. Elle peut intervenir seulement au cours de la résolution alors qu'un certain nombre d'opérations ont déjà été effectuées. Dans ce dernier cas, le sujet commence par exécuter certaines opérations réalisables étant donné l'état initial du problème et choisies à partir de critères autres que la considération du but à atteindre. A partir d'un certain point, correspondant à un état plus proche du but (en termes de nombre minimal d'opérations nécessaires) que l'état initial, il prend en considération le but à atteindre et, avant de poursuivre l'éxécution, détermine l'ensemble des opérations à réaliser. Dans le premier cas nous parlerons de planification préalable à l'éxécution, dans le second de planification en cours d'éxécution.

L'élaboration du plan peut être faite de façon régressive ou de façon progressive .

La planification régressive consiste à considérer le but, à définir quelles conditions doivent être remplies pour que le but puisse être réalisé, à examiner si ces conditions sont réalisables d'emblée ou si elles supposent elles mêmes d'autres conditions. Dans ce cas le sujet considère ces nouvelles conditions et procède pour elles au même examen, et ainsi de suite jusqu'à ce que toutes les conditions à prendre en considération soient réalisables à partir de la situation actuelle. Le plan des opérations à exécuter est alors déterminé en prenant les différentes conditions dans l'ordre inverse de celui où elles ont été examinées. On a alors un ordre (partiel ou total) sur l'ensemble des opérations à réaliser.

La planification progressive consiste à examiner quelles opérations sont réalisables à partir des données initiales, à anticiper quelles opérations nouvelles peuvent être faites à partir des précédentes et ainsi de suite jusqu'à ce que le but soit accessible. Le sujet examine quels sont les résultats effectivement utilisés pour le dernier calcul, conserve les opérations conduisant à ces résultats, et ainsi de suite... ; il ne retient en définitive que les opérations dont il utilise les résultats. Si cette procédure est appliquée systématiquement elle aboutit au même résultat qu'une prodédure régressive : il est donc difficile a priori de différencier empiriquement ces deux types de planification. Toutefois, si la planification n'intervient qu'en cours d'exécution, il est plus vraisemblable qu'il s'agit d'une procédure progressive. Une progression régressive en effet n'est pas plus facile à mettre en oeuvre après exécution de quelques opérations qu'avant toute exécution et il est plausible que, si le sujet se propose de procéder ainsi, il le fasse dès le début. Par contre, une procédure progressive peut être plus facile à mettre en oeuvre lorsque certains résultats ont déjà été obtenus : c'est le cas si ces résultats conduisent à un nouvel état du problème, où le but se trouve plus proche et où, par suite, la portée des prospections nécessaires pour anticiper le but peut être plus réduite.

3. - Facteurs étudiés.

Trois types de facteurs ont été pris en compte comme étant susceptibles de faire varier l'activité de planification.

- Les premiers sont relatifs à la structure du problème. Les problèmes ont été construits en faisant varier d'une part le nombre d'opérations nécessaires à la solution et d'autre part le degré de dépendance entre ces opérations. Il y a deux cas extrêmes : dépendance maximale et dépendance minimale. Dans le cas de dépendance maximale, chaque opération utilise le résultat de l'opération précédente ce qui définit un ordre total sur l'ensemble des opérations. Dans le cas d'une dépendance minimale, seule la dernière opération utilise les résultats des opérations précédentes. Pour elle seulement l'ordre d'exécution est contraint : les autres peuvent être réalisées dans un ordre quelconque. Des cas intermédiaires sont également étudiés dans lesquels certaines opérations dépendent du résultat d'une seule opération et d'autres dépendent du résultat de plusieurs opérations.

- Un deuxième type de facteurs est relatif à la tâche demandée au sujet. Dans la situation de référence le sujet dispose de toutes les données utiles pour déduire les informations nécessaires à l'obtention de la solution. Une autre situation a été proposée dans laquelle certaines données (utiles ou non) sont manquantes : le sujet doit demander celles qui sont nécessaires pour résoudre le problème et celles-là seulement. Cette seconde situation à été introduite avec un double objectif. Elle permet en premier lieu d'examiner si une activité de planification est plus facilement mise en oeuvre dans une situation où la planification devient beaucoup plus économique : en effet, le nombre de données manquantes est suffisament grand pour rendre fastidieux de faire toutes les demandes avant de rechercher si elles sont utiles. Elle permet par ailleurs une étude plus fine de l'activité de planification à partir de données comportementales supplémentaires : les données manquantes demandées, et surtout l'ordre dans lequel ces demandes sont faites. On peut ainsi mieux appréhender quels sont les calculs que se propose le sujet et l'ordre dans lequel il les envisage. On peut aussi mieux saisir si le sujet planifie dès le début ou ne commence à planifier qu'en cours d'exécution.

- Le troisième type de facteurs concerne les particularités de la situation expérimentale. Comme cela a été dit précédemment, deux classes de situation ont été envisagées : celles où le passage d'un état à un autre est une modification physique et celles où c'est la déduction d'une nouvelle information. A l'intérieur de ces classes plusieurs situations ont été construites, car les comportements de planification peuvent dépendre du degré de complexité des opérations à réaliser et des conditions d'exploration permettant de déterminer si les opérations sont réalisables et si elles sont pertinentes. Vraisemblablement ces facteurs n'ont pas une importance égale suivant le type de planification mis en oeuvre. Les particularités de la situation interviennent sans doute beaucoup plus dans le cas d'une planification progressive que dans le cas d'une planification régressive, car les possibilités d'anticipation, étant donné la charge mentale que cette activité implique, dépendent des caractères spécifiques de la situation, de la nature des opérations et des données, et de la configuration physique sous laquelle se présente le problème.

4. - Eléments de conclusion

. Dans le cas où les sujets disposent de toute l'information nécessaire à la solution du problème (1ère expérience) l'activité de planification se repère essentiellement par l'absence d'opérations inutiles dans les calculs effectués par le sujet. Pour toutes les situations, la fréquence des comportements de planification diminue quand le degré d'enchaînement des opérations augmente ; mais cette tendance n'a pas la même importance selon les situations. De plus les comportements de planification n'ont pas la même signification dans tous les cas.

Ils peuvent correspondre, selon les situations proposées au sujet soit à une planification de type regressif, soit à une planification de type partiel et progressif.

Les données recueillies sont compatibles avec l'idée qu'un nombre important de sujets mettent en oeuvre une planification progressive ; mais celle-ci ne permet de résoudre sans opérations inutiles, que les problèmes où le but est à une distance inférieure à la portée de l'anticipation : c'est le cas des problèmes où le degré de dépendance entre les opérations n'est pas très élevé.

. Dans le cas où il faut demander certaines informations, on constate une augmentation de la fréquence des cas où il n'y a pas d'opérations inutiles. Cette situation, où les opérations sont plus coûteuses dans la mesure où elles impliquent des demandes d'information, incite sans doute davantage les sujets à mettre en oeuvre plus complètement leurs capacités de planification (de type progressif ou regressif). Cette situation est certainement meilleure pour tester les capacités réelles de planification.

L'analyse des demandes d'information confirme pleinement les conclusions formulées à partir des premières expériences. La fréquence des cas manifestant de façon indiscutable la mise en jeu d'une procédure régressive est très faible. Les élèves observés font preuve dans leur majorité d'une procédure de planification partielle et de type progressif. Leurs choix initiaux sont basés pour une grande part sur l'exploration perceptive, comme en témoigne le fait que les premières demandes d'information concernent des positions très spécifiques.

. Ces résultats sont très importants du point de vue pédagogique dans la mesure où ils peuvent expliquer certaines observations sur la résolution de problèmes arithmétiques faites à partir des brouillons des élèves (COLOMB, RICHARD 1980). La découverte de la solution semble se faire par tâtonnements (essai de différentes opérations puis abandon au vu du résultat qui se révèle plausible), ceux-ci étant explicites ou purement mentaux. L'élève peut ainsi arriver à la solution par une logique qui ne repose pas sur une analyse de la structure du problème et retenir dans la rédaction définitive uniquement les opérations qui sont en fait utiles.

Les données sont compatibles avec l'idée qu'un nombre important de sujets mettent en oeuvre une planification progressive ; mais celle-ci ne permet de résoudre sans opérations inutiles, que les problèmes où le but est à une distance inférieure à la portée de l'anticipation : c'est le cas des problèmes où le degré de dépendance entre les opérations n'est pas très élevé.

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Ceci expliquerait la difficulté que représente pour les élèves, certains problèmes où il est pratiquement impossible d'aboutir à la solution si l'on ne déduit pas, à partir de l'analyse du problème, la relation qui permet de trouver la solution.

The Interaction Between Problem Structure and Problem Solving Processes

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Until recently most mathematical problem-solving researchers have been either assessing problem complexity as measured by success (Jerman, 1973; Jerman & Rees, 1972; Nesher, 1976), or exploring and screening the problem-solving processes that emerge (Kilpatrick, 1969; Lucas, 1974; Webb, 1979). The time has come to use our knowledge of problem complexity and problem structure in exploring the problem-solving processes and to use our knowledge of processes to enrich our assessment of problem complexity.

The purpose of the study from which this paper is reporting was two-fold: a) to use process and product variables to assess the complexity of word problems with given characteristics, b) to elicit a single heuristic process namely, guessing that leads to successive approximation, which will allow a detailed study of the particularities of such a process.

The focus in this report will be on the assessment of complexity using product and process variables and the knowledge gained by studying problem structure and relating it to the problem solving processes used by problem solvers.

The setting for the study.

This was a clinical study conducted with sixty, above average, seventh grade students, who had no background in algebra. Each of the students was presented with six word problems embodying all the systematic variations in structure and content that were intended in the study. Each subject was asked to think aloud as he/she attempted to solve each of the problems. Data was collected by three means: a) the written work of the student, b) the audio taping of the session, c) the written observation of the interviewer.

Analaysis of problem structure.

The problems were developed to embody a structural variable and a content variable. The content variable was determined by the number of unknowns and the number of conditions in a problem. The structural variable was the size of the search space of the problem.

The search space idea was developed by this writer to reflect the logical and numerical relationships between the numbers in a problem rather than the sizes of these numbers. It is similar to the corresponding concept in information processing in that it enumerates all possible moves that a subject may take while still abiding by a minimal set of rules. It is unique in that it is applied to numerical tasks rather than purely logical tasks. It was developed through the use of preliminary observations that indicated that problem-solving attempts were not completely random, even when they seemed to be so, and that in most cases attempts emerged from a domain of possibilities that satisfied some of the conditions in a problem but not necessarily all the conditions.

The definitions that follow will clarify the structural and content variables that were used in the development of the tasks.

> An unknown is a quantity described in the problem statement for which the value is not expressly given. A condition is a statement that can be translated into a linear equation involving at least two unknowns. The number of conditions in the problem is the number of linear equations that can be obtained by direct translation of the verbal problem statement. The search space of a condition of a problem with n unknowns is the set of all n-tuples of positive integers that satisfy the condition. A finite condition is a condition whose search space is finite. When all of the problem conditions are finite, the search space of the problem is the union of the search spaces of the conditions.

> To illustarte these definitions the following problem will be used: Jeff bought 5 jawbreakers and 10 pieces of bubble gum for $1.60. A jawbreaker and a bubble gum together cost 29¢. How much did Jeff pay for each piece?

Here the two unknowns are the price of a jawbreaker and the price of a bubble gum. The two conditions are the total price and the price of a jawbreaker and bubble gum together. The search space for the first condition is the set of ordered pairs of positive integers that satisfy the given condition, namely (2,15), (4,14), (6,13) ..., (30,1). The search space for the second condition is (1,28), (2,27),..., (28,1). The search space for the problem is the union of those two sets and its size is 42 elements or ordered pairs.

The problems incorporating those variables were six systems of simultaneous conditions whose characteristics are illustrated by the figure below.

	2 conditions 2 unknowns	2 conditions 3 unknowns	3 conditions 3 unknowns
Small Search Space	2 × 2	2 × 3	3 × 3
Large Search Space	2 × 2	2 × 3	3 × 3

Figure 1: Characteristics of the Six Systems of Simultaneous Equations Used

The six systems were embodied in six different stories. A list of the thirty-six problems and their search space analysis could be found in Harik 1979 and Harik 1981.

Problem Solving Processes Used by Subjects.

The problem solving processes observed were put into three categories namely: probabilistic moves, manipulative moves and certainty moves. Certainty moves were subdivided into four types. The following is a list of definitions of these categories:

> A move (or an attempt) is a series of purposeful actions, the result of which would lead the problem solver to a conclusion. The conclusion could be to abandon these actions, to modify them, or to maintain them when they produce satisfactory results (satisfactory results in the perspective of the problem solver).
>
> A probabilistic move is composed of the following elements: a guess on an unknown(s), calculations to verify that the guess satisfies the condition(s) in the problem, and a conclusion is made as to the result of the guess and the direction in which the guess should be modified.
>
> A certainty move is a move in which the final or preliminary result(s) is reached through the laws of deduction and/or induction.
>
> A deductive move type I is a certainty move that is derived by simple inference from given data.
>
> A deductive move type II is a certainty move that is derived by an inference through a process of elmination of variables as in solving simulatneous equations.
>
> A deductive move type III is a certainty move derived by a hypothetical inference made for the sake of estimation.
>
> An inductive move is a certainty move that detects a pattern of relationships among the variables and produces the pattern without the need to verify by calculations.
>
> A manipulative move is a series of manipulations of numbers and of operations in such a way that they can produce, in the problem solver's mind, a reasonable answer.

Assessment of complexity.

The attempt here was to assess the viability of three task variables as determinants of complexity.

The design was a multivariate, 2 × 3 factorial design with repeated measures on both factors. The factors being, the size of the search space (small vs large) and the type of information (two unknowns and two conditions, two unknowns and three conditions, three unknowns and three conditions).

Three dependent variables were used. The time taken for a subject to solve a problem, the performance of the subject on each problem and the number of moves or attempts made on each problem. The number of moves were determined after the processes were coded and classified into the categories mentioned earlier.

The conclusions are applicable to the population of average and above average seventh grade students who have not had any formal training in algebra when solving story problems embodying simultaneous conditions. Within the limitations set by the type of population and the type of content, the following conclusions can be made.

It was found that the size of the search space across all three types of information contributed significantly to difficulty levels of problems: the smaller the size of the search space, the easier the problem.

It was also found that the type of information given contributed significantly to the difficulty level of problems. When comparing for the amount of information given, problems with three unknowns and three conditions were easier than problems with two unknowns and two conditions. When comparing for the ratio of the number of unknowns to the number of conditions, problems with the same number of unknowns as conditions were easier than problems where the number of unknwons is greater than the number of conditions.

When attempting to interpret how a problem with three unknowns and three conditions can be easier than a problem with two unknowns and two conditions, it was necessary to look at the processes used by the subjects when solving those types of problems, as well as the type of conditions present in the problems. Problems with three variables and three conditions had two condtions where one is a subcondition of the other. Seventy-three percent of the subjects made a deductive move using these two conditions which effectively reduced the problems to subproblems with two variables and two conditions and very small search spaces. Furthermore, when looking at the dependent measures that reflected the significant difference in the difficulty levels, the number of moves was the major factor in determining the difference in the difficulty levels.

One may glean from these possible interpretations that the reducibility of a problem to subproblems and the associations of the problem and its subproblems to their respective search spaces may be a more important determinant of difficulty levels than the actual quantity of information involved in a problem.

The significant interaction between the size of the search space and the type of information given was mainly reflected by one dependent measure, the number of moves. The interaction indicates that the difference in the number of moves between problems 1 and 2* is greater than the difference in the number of moves between problems 5 and 6**. Here again, the size of the search space seems to contribute to the understanding of this interaction. Comparing the difference in the size of the search spaces of problems 1 and 2 (42 - 16 = 26) and the difference in the size of the search spaces of the subproblems of problems 5 and 6 (17 - 12 = 5), one may see how the amount of increase in the size of the search spaces contributes to the increase in difficulty level.

Looking at all of the above results, it is reasonable to conclude that the notion of the search space and its size plays a crucial role in determining the difficulty level of problems within the given population. The use of process measures as well as product measures to assess difficulty provided interpretations that would not have been possible if only product measures were used. Also, the use of multivariate analysis provided different and more accurate information than the use of univariate analysis.

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*Problems 1 and 2 varied in the size of their search spaces but both had two unknowns and two conditions.

**Problems 5 and 6 varied in the size of their search spaces but both had three unknowns and three conditions.

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DIMENSIONS OF DIFFICULTY IN SOLVING GEOMETRICAL PROBLEMS

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Uwe Lehnert, Freie Universitaet Berlin
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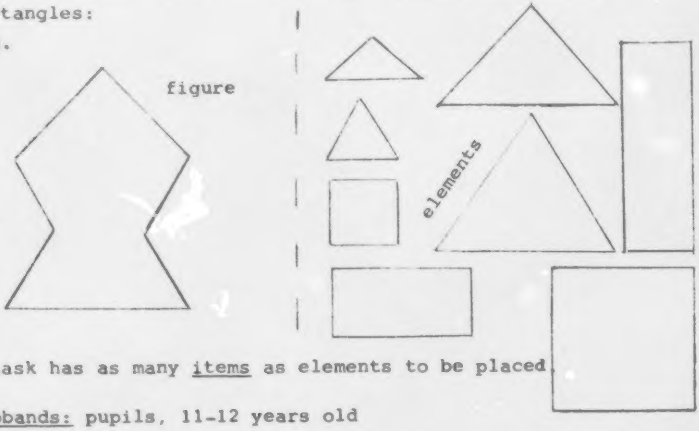
Conception and results of an empirical investigation.

Material: "Formenspiel matema" (Bauersfeld-Kleinschmidt)

Task or problem: a given geometrical figure is to be covered by elements of the matema-material: triangles, squares and rectangles:

e.g.



A task has as many items as elements to be placed.

Probands: pupils, 11-12 years old

Purpose of the investigation: exploration and definition of

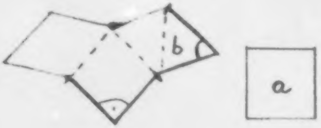
a) dimensions (factors) of difficulty of the items

b) dimensions (factors) of ability the pupils needed to solve items

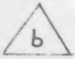
The result we got in former experiments made us assume that the difficulty of an item can be divided into different dimensions of difficulty, which are determined by psychological operations used in solving the item.

The exact knowledge of these dimensions would help to construct sequences of lessons training the solving of similar problems. We tried to define these dimensions theoretically. We thought that angles and sides indicated by the figure would determine the dimensions of difficulty.

e.g.



there is an indication in the figure of two sides, one angle and one point for the position of the square a



there is an indication of two sides and one angle for the position of the triangle b

We thought it would be essential to know how many parts of the element to be placed are indicated by the figure [in proportion to the number of defined parts of the element; 3(sides) + 3(angles) = 6]. But when we observed children solving the problems, we didn't find our theory verified. There seem to be additional dimensions of difficulty in the mind of children. But the pupils couldn't define these dimensions of difficulty because they couldn't perceive them seperately.

Methodology: searching for a method to define these dimensions of difficulty, we found two methods used in social sciences:

"clustering" and

"metric and non-metric multidimensional scaling, MDS".

(D. Steinhausen/K. Langer, 1977; Gerd Gigerenzer, 1981)

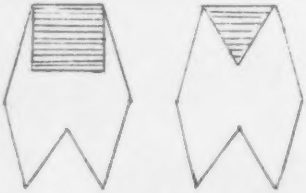
We became interested in comparing the results obtained by these procedures with the estimation of the difficulty of items by "linear logistical models" (Kluwe/Spada, 1981).

This presentation will only describe how we tried to analyse the dimension of difficulty by the MDS-procedure.

1. The matema-material was explained by the teacher. The pupils solved some problems. (2 lessons)
2. Test: A team (of 2 professors, 6 students) observed 26 pupils, 11-12 years old, at solving 24 problems (Formenspiel matema: alef). Each pupil was observed by one professor or student. The procedure of solving was recorded in detail.

   The instruction given to the pupils was: "The given figure must be covered by elements of the matema-material. Use at each step elements as large as possible!"

   When a pupil placed an element that would have hindered the "parquetting", the professor or student replaced it by the correct one.



   Each of the problems had as many items as there were elements to be placed into the figure. The pupil wasn't allowed to move the element after it had been put down.
3. The analysis of the investigation was done by computer.

   We used MDS(X) program written by Roskam and Lingoes
   in the Edinburgh version
   and a special program written by W. Reitberger, Berlin.

   It wasn't possible to analyse the solving of a problem as a whole, because the strategies of solving applied by the pupils differed considerably from one another. We therefore decided to analyse the single items constituating the problem.

   We only considered items which had been solved by 15 pupils at least and in the solution of which at least 5 mistakes had been made. We thus analysed 16 items (see appendix 1).

The different items a pupil had tried to solve were compared to one another (criterion: solved - unsolved) and the pair that was thus compared got the index 0, if the solutions of the items corresponded to one another, the index 1, if they didn't.
This analysis was done for all pupils. Then we added up the non-conformity for each pair of items considering all the pupils.
This number of non-conformity was divided by the number of pupils who had tried to solve both of the items.
There had to be more than 8 pupils who tried to solve both of the items, otherwise we didn't consider this pair of items.
We thus got a matrix of "non-conformity". We interpreted this matrix as a matrix of "dissimilarities" of the items (see appendix 2).

The procedure of MDS (minissa) gives an interpretation of the dissimilarities as distances (Kruskal, 1964) and illustrates these distances as points in a space with n dimensions. We tried it for n=1 to 5, but then we decided: n=3, because of the stress. (The stress is a measure for goodness of solution, i.e. the correct distribution of the items in a multidimensional space.) For n=3 the stress was 0,10505 (see appendix 3).
We also tried the metric version of the MDS-procedure. Between the two versions there were substantial differences. We then considered the mistakes that had been made in the solution of the items (see appendix 4).
We compared some items in extreme positions.

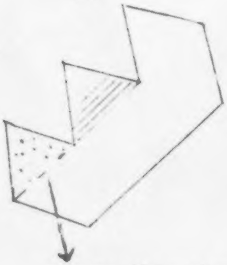
Result : We found reasons to assume the following interpretation for the three dimensions:
Dimension 1 : in this dimension the essential differences between items in extreme positions can be explained by the grade in which the given figures leed astray, make the pupils place a wrong element.

"grade of leeding astray"

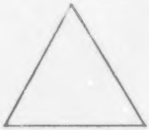
(-) ______________________________ (+)

e.g.

a)

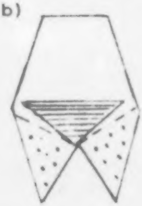


the pupil placed the little
the correct element to be placed
would be the large triangle



means: this element is already placed, the task for the pupil is to continue "parquetting"

b)



the pupil placed the little rectangular triangle assuming an angle of 90°;

the correct element to be placed would be the little equilateral triangle

Dimension 2: in this dimension, the essential differences between the items in extreme positions can be explained by the number of parts of elements (angles, sides, ...) indicated by the figure.

"grade of determination"

(-) ______________________________ (+)

(according to our theory)

Dimension 3: in this dimension, the essential differences between the items in extreme positions can be explained by increasing attention paid by the pupils to the angles indicated by the figures – or the angles they wrongly assumed in the figures

"grade of attention paid to the angles"

(-) ______________________________ (+)

In the inverted direction, this dimension is determined by the "grade of attention paid to the sides".

Further intensions:

a) We will try to verify this result by testing further 60 pupils.

b) We will try to compare the pupils to one another, who had tried to solve each item. We thereby will get a matrix of dissimilarities of the pupils to be analysed by a MDS-procedure.

c) We will try to estimate the difficulty of the items and the ability of the pupils by "linear logistic models" in order to compare the results.

d) We will try to construct a curriculum to train the pupils in solving similar problems, considering the obtained dimensions of difficulty.

e) We will make an evaluation of the curriculum.

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HOW DO STUDENTS PROCEED IN PROBLEM SOLVING

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Abstract

There are students and there are problems where the students ...

(a) ... use unexpected guess-and-test procedures,

(b) ... replace-after a certain amount of failures-the original problem with an easier one.

We see two reasons for that behaviour:

(a) The guess-and-test procedures help to build up a relational understanding of the problem.

(b) Having success is more important for the student than having a correct solution of the original problem.

This paper will describe some of our experiments. The presentation at the conference will include selected portions of video-taped problem solving situations to substantiate our observations and our conclusions.

1. Calculator Experiments

The key experiments originate from our calculator research. Calculators facilitate guess-and-test strategies since it is very easy to test an assumption by pressing the appropriate buttons and comparing the calculator display with the suggested answer. Our first experiment was discovered by chance. The second experiment belongs to an extensive sequence of calculator games.

1.1 Discover your calculator

We use a calculator where the buttons have no description (or symbols) on their surface. That means the calculator works as any other one whenever you press a button. But to use the calculator meaningfully you first have to find out the mode of operation of each button: Where are the digits, where the basic operations, where the equal button, etc.?

First we present a couple of different calculators to the test person which have symbols on the buttons to make sure that the test person has all the knowledge needed for our experiment. Then the test person gets the experimental calculator (with 24 buttons) to find out the mode of operation of a specific button (it is the $\boxed{M-}$ -button). After the experiment the test person was asked to describe what he did.

Observations: In every experiment the subjects (age 20 to 40) pressed different buttons more than 200 times. There were experiments with more than 1000 key strokes. There were a lot of "meaningless" repetitions, i.e. $\boxed{C}\boxed{5}\boxed{+}\boxed{2}\boxed{=}$ or $\boxed{C}\boxed{5}\boxed{+}\boxed{2}\boxed{=}\boxed{M-}\boxed{=}$. Every subject estimated the amount of his key strokes three to ten times smaller than it really had been.

1.2 Hit the target

The students (age 11 to 25) have to find by guess-and-test an appropriate second factor in $a \times \boxed{?} \longrightarrow [b, b+c]$ to get a product p with $b \leq p \leq b + c$ (a,b and b+c are given numbers). They keep a record of their guesses and according displays.

Observations: The students developed their strategies mostly intuitively. There were many partial strategies and more guesses than necessary.

2. Problem Solving

We found similarities of the student's behaviour also in other problem solving situations, i.e. percentage problems, solving equations, working on complex shopping problems, etc. But these video-taped situations did not show enough specific behaviour worth to demonstrate. Therefore we decided to use more manipulatives. This led us to geometry problems.

3. Plane Geometry

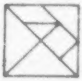
3.1 Puzzles

The subjects (age 20 to 25) had to work on different puzzles.

Observations: They all worked only on a subset of attributes (colour, frame, shape, ...) and used guess-and-test procedures for the remaining (forgotten) attributes.

3.2 Tangram

The subjects (age 20 to 25) had to build fig. 2 out of the seven pieces of fig. 1.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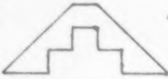
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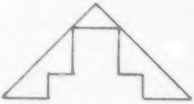
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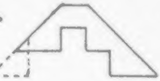
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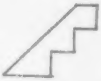
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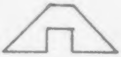

(6)

Observations: There were often replacements of the original problem (fig. 3 or fig. 4) or many repetitions of easier problems (fig. 5 or fig. 6).

3.3 Tesselation

The subjects (age 20 to 25) had to decide which of the given tesselation plates would tesselate completely which of the given areas by writing "yes" or "no" in an appropriate table. The students could use the manipulatives but they were not forced to use them.

Observations: Many mistakes in the tables (not all attributes were used to make a decision). 3 of the areas had straight lines as a boundary, one typ of the tesselation plates was a bird (Escher configuration). Every student first tried out if the birds fitted.

4. Threedimensional Geometry

4.1 Puzzles

The subjects (age 20 to 25) received 12 wooden pieces to build up a cube. (Other puzzles of a similar type present a ball or an egg).

Observations: Several times one piece was left over. The student hid the piece and closed the holes in the cube with his fingers.

4.2 Rubic's Cube

Observations: Many, many "meaningless" repetitions (age 10-40)

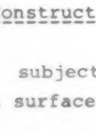
4.3 Counting surfaces

The subjects (age 11 to 40) received 4 solids (cube, octahedron, dodecahedron, icosahedron) to count the amount of surfaces of each solid.

Observations: All subjects started counting immediately. Many failures were necessary to experience a need for a counting strategy.

4.4 Drawing a solid

(7)



Draw a given solid (see fig. 7) (age 11).

Observations: Many trials, erased with the rubber. Carefully drawings of straight lines. Many measurements without using the measurement results for the drawing.

4.5 Constructing solids

The subjects (age 11) had to construct given solids out of given surface pieces (or out of straw).

Observations: Many repetitions in building sub-parts of the solids.

5. Summary

There seems to be a general "structure of behaviour" when students solve complex problems. They are so strongly goal-oriented that they do not use all the knowledge and abilities they really have. Instead they try to internalize the problem by mostly unconscious guess-and-test (i.e., trial-and-error) procedures. If they have too little success, they replace the original problem with an easier one and work on that one to attain more success and thereby build self-confidence. The solution of the original problem becomes possible only if their self-confidence has been built up to such an extent that they feel free to leave the guess-and-test mode.

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C. LANGUAGE

THE ASSESSMENT OF ORAL PERFORMANCE IN MATHEMATICS

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Usually, written tests are used to assess the performance of students. But it is often demanded by parents as well as by school authorities that this should be complemented by assessing the oral performance. So the Ministry of Cultural Affairs of Baden-Württemberg has issued an order according to which

- assessments of oral performances have to be considered for the final marks in the school reports,
- the criteria which are decisive for the assessment of oral performance, have to be made known in the beginning of the school year.

A statement which explains what "oral performance" in that order means, is contained in a comment by Elser (1976). Thereafter

- the assessment must not refer to an unspecified general impression,
- the mere interest in the lessons (putting his/her hand up) is not decisive,
- the contents, the quality of the contributions is decisive.

The government demands to assess the quality of oral contributions to lessons. What do teachers do? A small questioning with the responses of 43 mathematics teachers being evaluated, showed that in spite of the order

- 4 of those 43 did not assess oral performance,
- 5 assessed only a general impression,
- 1 assessed the general impression and the frequency of putting the hand up and nothing else.

The usage at schools is illustrated by the answers to the question: What do you assess if you give marks for oral performance? The choice of answers was:

General impression	26
Frequently putting his/her hand up	8
Home work	12
Working out home work on the blackboard	9
Working out on the blackboard (other problems)	27
Questioning of rules, laws, etc.	19
Other kinds of oral performance	17

Those 17 teachers specified "other kinds of oral performance" as:

Suggestions for solutions	13
Right answers	4
Correction of statements	3
Quality of contributions	2
Thinking efforts	2

It is interesting that 24 teachers felt sure, 15 unsure in assessing oral performance.

These answers show only part of the problems with the assessment of oral performance. Many mathematics teachers feel helpless and left alone with this duty. In books on education almost nothing is to be found on this subject. So the question arises if oral performance should really be assessed and which the criteria for an assessment might be.

Schröter (1981) gives some arguments for assessing oral performance:

- Students contribute by zealousness, good ideas,etc. to lively and interesting lessons. These contributions should not be dismissed as unimportant for the assessment of performance.
- The ability to represent and defend positions in oral speech should be developed and therefore also be assessed.
- Speaking activities in lessons would in the feelings of many students be estimated as less important if oral performance is not assessed.
- But it should also be considered that there are students who follow the teaching quietly and tacitly, then dispose of recallable knowledge and show good written performance. It should be tried to get them to oral contributions.
- There is many a student with an unbalanced quality of oral and written performance. It would be unjust to favour part of the students by assessing only written performance.

The arguments are in favor of assessing oral performance. But I think it should be carried out in mathematics teaching only if it covers a field which is not or only insufficiently covered by assessing written performance. To decide upon this problem we have to trace the question to which achievements mathematics teaching should get the students, which are the goals of mathematics teaching.

After Wittmann (1978), p.47, three kinds of goals can be distinguished:

- Knowledge of facts, rules, laws, theorems
- Intellectual techniques (algorithmic thinking, knowledge of proce-

dures)
- Cognitive strategies (mathematizing, creativity, ability of reasoning)

In written class tests the achievement of goals of the first two kinds can be well examined, for almost only routine problems are given:
- Solution methods are to be applied to problems which have been solved in identical or almost identical form.
- If mathematizations or reasonings are required they have previously been carried out and practised in the lessons.
- The performance consists in the reproduction of facts, reasonings, and solution procedures.

These routine problems are to be accomplished in a given time, almost no space is admitted for creative thinking. So in written tests it is not examined how far the advancement is to the important goals of the development of cognitive strategies. Therefore the assessment of oral performance in mathematics should mainly concentrate on the progress in this field, if it should reasonably complement the assessment of written performance. But this requires that the lessons are organized in such a way that these aspects are sufficiently promoted. The traditional way of teaching mathematics was (and is still) often just limited to methods of storing recallable knowledge, drill of computational skills, left little or no space for creativity, and restricted itself in regard of mathematizing and reasoning to demonstrations by the teacher which the students had to reproduce. In this kind of teaching it should be abstained from assessing oral performance. It should only take place in problem oriented teaching with enough space for developing cognitive strategies.

Criteria for the assessment of oral performance can be obtained by further differentiation of the goals of mathematizing, reasoning, and creativity, such as e.g. in Wittmann (1978), p.48/49. Some of them are listed below.
- Finding mathematical models for situations.
- Getting, checking, and processing data.
- Interpreting adequately solutions, results.
- Supposing and finding solution procedures and proofs.
- Recognizing and formulating generalizations.
- Going beyond given informations.
- Expressing of and giving reasons for mathematical thoughts clearly and correctly.

- Checking statements and reasonings.
- Keeping to agreements (e.g. definitions, rules).

Beside observations of these abilities and attitudes the assessment of oral performance can also take into account things relating to knowledge of facts and to intellectual techniques, e.g. disposing and applying of definitions, rules,theorems.

Such a catalogue of criteria is not to be understood in the way that a teacher has to go through it completely at each assessment. In any concrete case it suffices to consider a few criteria. The catalogue should be an aid in getting a differentiated picture of the oral performance. Its application makes in any case high demands on the teacher. As the questioning of the teachers showed, do not many of them dispose of a differentiated catalogue of criteria.

The time of the assessment is an important problem, too. It should not take place during the teaching, because only a negative influence on the activities of the students is expected if the teacher teaches with a note book for marks in his hand. But it should be noted while the impressions are fresh, immediately after the lessons. I propose, the teacher should choose at random two or three students before each lesson and observe them in particular. This does not exclude that he may also assess remarkable performances of other students. I know teachers who successfully carry out a procedure like this.

Assessments of oral performances are much more determined by subjective perception and estimation than those of written performances. It must be taken into account that they do not always justice to criteria of objectiveness and that they are influenced by the well known tendencies of misjudgement (preinformation on the student, primacy/recency effects, etc.). Therefore assessments of oral performance, even if important goals are involved, should not be weighted as high as those of written performance. In my opinion at most a third is a reasonable portion for them. With this weight they influence a final mark only if the assessment of oral performance differs from that of written performance by more than one. This corresponds to the usage of many teachers.

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FIRST GRADERS' SOLUTION PROCESSES IN ELEMENTARY WORD PROBLEMS

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In this contribution we will report concisely the main findings of two studies, in which some aspects of the model of arithmetic problem solving developed in the late seventies by Greeno, Heller and Riley with respect to elementary addition and subtraction problems were tested (Greeno, 1980; Heller & Greeno, 1978; Riley & Greeno, 1978). Greeno and his coworkers argue that previous models of problem solving, such as the model developed by Bobrow (1968), do not give an appropriate description of the solution processes of competent problem solvers. Bobrow's model is a syntactic one, in which the main process is translating the verbal text into equations. In the model developed by Greeno and his coworkers, semantic processing is considered to be the crucial component in skilled problem solving. According to that model, the problem solver constructs a semantic representation of the relations between the quantities in a simple addition and subtraction problem in terms of one of three different semantic schemata - namely, a cause/change schema, a combination schema, or a comparison schema. The cause/change schema is used for understanding situations in which some event changes the value of a quantity; for example : "Joe has 3 marbles; Tom gives him 5 more marbles; how many marbles does Joe have now ?". The combination schema is "used to represent situations where there are two amounts, and they are considered either separately or in combination", as in the following case : "Joe has 3 marbles; Tom has 5 marbles; how many marbles do they have altogether ?". The compare schema "involves two amounts that are compared and a difference between them", such as this problem : "Joe has 3 marbles; Tom has 5 more marbles than Joe; how many marbles does Tom have ?" (Greeno, 1980, p.9 - 11 and 14). On the basis of the semantic schema, the problem solver then selects and executes an arithmetic operation, without the intervention of syntactic processes to translate the text of the problem into a mathematical equation. The arithmetic operation - an addition or a subtraction with the two given numbers - is sometimes associated directly with the semantic schema; in some cases, the operation is selected after a transformation of the original schema - for example, a cause/change schema - to a combination structure. The selection of the operation - addition or subtraction - as well as the necessity of a transformation of the original schema to a combination schema depend upon the problem type. Greeno and his coworkers have distinguished 14 problem types based on the combination of three characteristics of word problems :

(1) the semantic structure; (2) the identity of the unknown value : in a change problem, for example the unknown can be the starting value, the change quantity, or the result; (3) the direction of the event or the relation, such as increase or decrease in a change problem. According to Greeno the representation of the problem : "Joe has 2 marbles; Tom gives him 4 more marbles; how many marbles does Joe have now ?" will be associated directly with the addition 2 + 4. On the other hand the problem : "Joe has 2 marbles; Tom gives him some more marbles; now Joe has 7 marbles; how many marbles did Tom give him ?" requires a transformation from a cause/change to a combination schema before the appropriate operation is found.

Greeno c.s. developed their model of arithmetic word problem solving in the form of a running computer program, and, until recently, empirical data with respect to the model were scarce. In two investigations with first graders we have collected empirical data with respect to the Greeno-model. In one study, ten high-ability and ten low-ability first graders were given nine word problems near the end of the school year. During an individual interview, they were asked the following with respect to each problem : (1) to read it aloud; (2) to retell the story; (3) to solve it; (4) to explain and justify their solution method; (5) to write a matching number sentence; and (6) to build a material representation of the story using puppets and blocks. A second study was done with thirty children who had just entered the first grade. They were given seven word problems during an individual interview. The same general procedure was applied as in the preceding study with respect to each problem. But,in step (1) the interviewer read the problem text aloud, and step (5) was deleted.

The first study has yielded several results that support Greeno's basic hypothesis concerning the importance of semantic processing as a component of skilled problem solution.

First of all we found differences in the levels of difficulty of problems representing the three semantic schemata. Our data are in line with results obtained by Riley & Greeno (1978) : in general, cause/change problems are easier than combine problems, which are themselves less difficult than compare problems.

More important than those quantitative results, in our view, are the qualitative data that bear upon children's actions in response to the different tasks during the individual interview. Those data provide more direct evidence to the hypothesis that skilled problem solvers construct a representation of the semantic relations between the quantities in the problem situation. Indeed, in the solution processes and the actions of the high-ability subjects, we discovered a considerable number of elements that are indicative of semantic processing, while the low-ability subjects produced symptoms of a more superficial, syntactic solution procedure. We can mention only briefly some findings relating to the following tasks which were performed during the interview : (1) retelling the word problems; (2) writing a matching number sentence; and (3) justifying the arithmetic operation.

(1) Underlying the retelling task is the assumption that the degree of the child's success and the way in which (s)he retells the story yield information about the pupil's internal representation of the problem. We accepted the retold story as being correct when it contained the same semantic and mathematical structure as the original problem text. In general, the high-ability (H) group performed much better - that is, 85 % correct - on this task than the low-ability (L) group which achieved less than 50 % correct. For example, the first problem : "Tom has 2 apples; Els gives him 5 more apples; how many apples does Tom have now ?" was retold correctly by nine of the ten H-subjects, while only three of the ten L-subjects were successful.It is noteworthy in this respect that, especially in the correct performances of the H-group, there was often no verbatim correspondence between the original text and the retold story. For example, a H-child retold the first problem mentioned above as follows : "Tom has 2 apples; and he gets 5 more apples; then they ask how many apples he does have". It is interesting to compare this to an incorrect answer which came from a L-child : "Tom has 2 apples; he gives Els 5 more apples; how many apples does Els now have?". We consider the correct results elicited from the H-group on the retelling task as evidence of a semantic approach to the word problems. The findings that the L-pupils were much less able to retell the stories appropriately points to the absence of such semantic processing.

(2) A syntactic model predicts that word problems are solved by translating the verbal text into the corresponding mathematical equation which is, then,figured out. During the individual interview the children were asked to explain their solution methods. It was found here that only very rarely did a pupil mention the number sentence which best matches the word problem.
They often stated another number sentence or they said that they had counted either internally or on their fingers or that they knew the answer from memory. In this case, we still asked that the number sentence that best matches the story problem be written. Many pupils had great difficulties in completing this task successfully, even children who had just previously given a correct answer. As an example, reference is made here to the fourth problem : "Jan has 7 apples; he gives some apples to Els; now Jan has 2 apples; how many apples did he give to Els ?". Four H-pupils were unable to write the best matching number sentence. One of them wrote, "7 + 2 = 9"; the others, who had solved the problem correctly mentioned the arithmetic operation that they actually used to solve the problem - "7 - 2 = 5" or "5 + 2 = 7" - instead of the number sentence that would be derived

from a syntactic procedure such as "7 - . = 2 or 7 - 5 = 2". Most pupils who answered by giving one of those latter number sentences needed considerable time to perform the task; they reread the word problem entirely and constructed the sentence gradually. In other words, they applied a syntactic procedure, but their activity showed that they conceived the assignment as a new task and not at all as a way of making their solution procedure explicit. Those findings obviously show that skilled problem solvers often solve word problems correctly without transforming the verbal text to a mathematical equation. Those results, therefore, can also be construed as supporting a semantic model of the solution process.

[illegible]) The pupils were also asked to justify their solution procedures, especially their selection of the arithmetic operation for each problem. A comparison of the L and H subjects reveals an interesting difference between the two groups.
A typical reaction from L-learners was referring to isolated words in the problem text with which some arithmetic operation is associated : for example, the words, "more" and "altogether", are associated with adding, the words,"less" and "loose", with subtracting). They apply what is called a "key-word" strategy, that is indicative of syntactic processing. The H-learners on the contrary, more frequently use the total content of the story in their justification. An illustration relates to the following problem : "Tom and Ann have 8 apples altogether; Tom has 2 apples; how many apples does Ann have ?".
One H-learner justified his selection of the operation 8 - 2 to solve the problem as follows : "Tom and Ann have 8 apples altogether, and Tom still has 2 of them".
One L-learner solved the problem by counting on his fingers - first counting 8 and then further on counting 10. In justification he said, "Altogether, that means adding !".

The preceding results support the basic hypothesis of the Greeno-model of arithmetic word problem solving : skilled problem solvers seem to base their selection of an arithmetic operation on a semantic representation of the story problem rather than on a syntactic procedure. However, we have also found results, especially in the second study with beginning first graders, that raise questions with respect to some aspects of the theory advanced by Greeno c.s.. Here we will mention two critical remarks briefly.

(1) In their model Greeno c.s. attribute a crucial role to the combine-schema. They presume that, out of the 14 problem structures that they distinguish, only six are directly associated with the arithmetic operations of addition or subtraction. In the other cases such as cause/change problems in which the starting value or the change quantity is unknown and most of the compare problems as well, a transformation to a combine-schema which is then associated with one of the

operations is required. Greeno himself (1980, p.14-15) has stated that this part of the theory is largely speculative and that some of his early findings cast doubt on the assumption; our data strongly support this doubt. During the individual interviews in our two studies, it became apparent that only a fraction of the H-learners solve the problems in a way that is in accordance with the model. The following problem may serve as an illustration : "Tom has 2 beads; Ann gives him some more beads; now Tom has 8 beads; how many beads did Ann give him ?". Many children, some from the H-group as well, did not solve this problem by subtracting the smaller number from the larger one as the Greeno-model predicts (Greeno, 1980, p.16), but they count upward from the smaller number until they reach the larger one, giving the number of units counted as the answer.

We discovered another typical strategy with respect to compare problems - namely, the matching strategy : the child puts out two sets of cubes corresponding to the two given numbers; the sets are then matched one-to-one, and the answer is found by counting the unmatched blocks (Carpenter & Moser, 1982, p.18). There is neither empirical nor rational evidence to allow the assumption that children who solve those problems by means of such alternative, more informal strategies first transform the tasks to a combination schema. It should be added that, in more recent work, Riley, Greeno & Heller (in press) have abandoned their original position with respect to this point. In that paper, they distinguish, besides the semantic schemata, two other kinds of knowledge which are employed during problem solving : different kinds of action schemata and strategic knowledge for planning solutions. The solution process is conceived as follows : a semantic representation of the problem situation is constructed; planning procedures then use action schemata to generate a solution to the problem. This new version of the theory is certainly more compatible with the available data.

(2) According to Greeno c.s., the three semantic schemata are necessary and sufficient for understanding all simple addition and subtraction word problems. As stated before, we endorse their viewpoint concerning the importance of semantic processing as a component of skilled solution processes. However, some of our data suggest that, besides the semantic schemata, the problem solver must also command a more general schema that involves the external and internal structure and the role and intent of word problems; the problem solver activates this word problem schema whenever he is confronted with such tasks. The following data can be considered to be evidence for the reality and importance of the word problem schema. In reacting to word problems, some L-learners in our studies, especially younger children, did not show any solution activity, while the actions and answers of others were inappropriate. In the study with beginning first graders, the following problem was presented : "Pete has 3 apples; Ann has

also some apples; Pete and Ann have 9 apples altogether; how many apples does Ann have ?". Several children answered, "some apples", "a couple", or "a little". Another example is taken from an answer protocol of a collective test in an earlier study and bears upon the following problem : "In the evening Charles has 6 marbles; during the day he lost 2 marbles; in the morning Charles had...". One pupil did not answer with a number, but wrote, "played with them".
It seems to us that, in the Greeno-model, mastery of the general word problem schema is implicitly assumed. This assumption is probably related to the reality that construction of the model was based mainly on computer simulation. We argue that, in view of a comprehensive theory of word problem solving, it would be worthwhile to study the nature and development of the general word problem schema as a component of the equipment of a skilled problem solver more systematically in future research.

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RECENT STUDIES IN STRUCTURAL LEARNING

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The research described here is a continuation of work begun by Dienes and Jeeves (see Jeeves, 1971 for a summary). Tasks based on set operations and group structures have been used to study the sort of learning which is made possible by the detection and exploitation of structure in the material to be learned. These tasks follow the standard pattern of a series of trials, on each of which a stimulus (a pair of sets, or a pair of elements) is presented by the experimenter (reception paradigm) or selected by the subject (selection paradigm). After a prediction has been obtained from the subject, the correct response (the outcome of applying the set or group operation) is displayed. Trials continue until the subject has demonstrated learning by reaching a criterion of a certain number of consecutive correct predictions. Three findings of theoretical interest are reported here.

Transfer between isomorphic tasks

One way of investigating precisely what has been learnt in a learning task is to follow it with one or more tasks structurally related to the first; a special case is where the tasks are isomorphic. Halford (1975) showed that adult subjects tested on a series of tasks all based on the 3-group, but using different symbols, could transfer with almost optimal efficiency. In an unpublished study carried out in Argentina, children were tested on two tasks based on the 2-group. In the first, they had to learn to predict outcomes as follows: Yellow + Yellow = Yellow, Yellow + Red = Red, Red + Yellow = Red, Red + Red = Yellow; the second task used shapes. For 9-year-olds (N=20), the mean numbers of errors to criterion on the two tasks were 11.8 and 9.2 respectively; for 13-year-olds (N=40), they were 9.3 and 4.7. Thus there is a marked interaction in that, while the groups differed little in performance on the first task, the older children, but not the younger ones, showed substantial improvement on the second. In a follow-up experiment using three 2-group tasks, Owens (1979) got similar results with three groups of 16 children, the means for errors to criterion for tasks 1-3 respectively being: 9-year-olds 10.5, 10.1, 10.6; 11-year-olds 9.4, 9.9, 7.9; 15-year-olds 10.5, 8.7, 6.7. The 2-group task involves only four combinations, so it can be tackled by rote memory; alternatively, some rule can be worked out. The suggested interpretation of the results is that the younger children predominantly relied on rote memory, which provides no basis for transfer, while the older ones, or at least a substantial proportion of them, exploited the isomorphism in some way, such as by encoding a general rule. This interpretation tallies with that of Somerville and Wellman (1978) who tested children aged between 10 and 14 on a task based on the cyclic 4-group. They concluded from their results that "the best present interpretation ... seems to be that many of the youngest children use a memorization strategy and use it fairly efficiently, while the oldest children use a strategy of trying to apprehend some system for the items and use it fairly efficiently" (p.84). If these interpretations are correct, these findings point to an important developmental change, namely the emergence of a readiness to look for structure in data.

The relationship between difficulty and complexity (Greer, 1979)

An apparatus was constructed to represent set operations. It has two groups of lights (red, green, yellow, white and blue) at the top, some of which are switched on, to represent two sets. A third group of lights is at the bottom; by pressing a switch, those lights come on which represent the result of applying whatever set operation has been programmed, to the sets selected at the top. The operations used were Union (U), Intersection (I), Symmetric Difference (S), Difference (D) and their complements (U',I',S',D'). The tasks involved working out the rules corresponding to these set operations from a series of instances. In the first experiment, subjects aged between 8 and 15 (N=179) were tested on one operation each, using a reception paradigm.

Expected age effects were found, and by regression analysis it was possible to place the operations on a scale of difficulty as follows: U(0) I(.26) U'(.38) I'(.40) S(.47) S'(.73) D(.85) D'(1). In the second experiment, subjects aged between 9 and 15 (N=64) were allowed to choose instances themselves and were tested on 8 different rules in succession (using a Latin Square design). Again the expected age effects were found, and a similar difficulty scale computed as follows: U(0) I(.22) U'(.36) S(.41) I'(.54) S'(.59) D(.63) D'(1) (r=.95 with previous results).

What underlies this pattern of difficulty? In the second experiment, subjects were asked to describe the rules when worked out. By counting the number of words in these descriptions, a "codability" scale was worked out for the rules using regression analysis. The correlation between this scale and the difficulty scale was .93. A structural analysis of the descriptions showed that three levels of complexity could be distinguished: 1. U and U' were mainly described in terms of whether or not the lights were on at the top, without qualification 2. I, I', S and S' were described in terms of the number of lights of a given colour on at the top 3. D and D' were described in many different ways, all of which were more complex than descriptions at the first two levels. These levels account for most of the observed order of difficulty among the rules, if complementation is assumed to increase the difficulty somewhat.

We have established a link between complexity and difficulty of learning rules; in contrast to many previous efforts, however, the measure of complexity has been based on descriptions elicited from subjects rather than on some a priori basis (such as Information Theory). As Simon (1972) states: "if an index of complexity is to have significance for psychology, then the encoding system itself must have some kind of psychological basis" (p.371). Thus we have provided some empirical evidence relevant to the philosophical question of the connection between complexity and difficulty. I will not tackle here, however, the extremely difficult question as to why some rules have shorter descriptions than others, when expressed in natural language.

Alternative descriptions of the same structure

In discussing experiments on discovery of patterns in sequences, Simon (1972) also said: "It is quixotic to seek crucial experiments to determine the coding language, because ... different subjects almost certainly use different representations for the same sequences ... Instead of seeking to discover the pattern code or the complexity measure, we need to discover the conditions ... under which subjects will adopt one or another coding scheme ..." (p.382).

Consider the Klein 4-group represented thus:

FIRST SYMBOL

		○	●	△	▲
	○	○	●	△	▲
SECOND SYMBOL	●	●	○	▲	△
	△	△	▲	○	●
	▲	▲	△	●	○

□ = yellow

■ = red

Two of the ways of describing this are: 1. Relational: If the two symbols are the same, the result is the yellow circle; if they are the same shape but different colours, the result is the red circle etc. 2. Operational: If the second symbol is the yellow circle, the outcome is the same as the first symbol; if the second symbol is the red circle, the outcome is the same shape as the first symbol, but different in colour etc.

Of 106 students tested on a task using this structure, 26 came up with a relational description, and 4 with an operational description, when asked to describe the structure. The task itself was in two phases - a learning phase during which subjects went through a standard series of trials until criterion or a maximum number of trials was reached, and a test phase during which subjects who had reached criterion on the learning phase had to retrieve outcomes as quickly as possible, the latencies being measured. The classification of verbal descriptions of the structure as relational or operational was found to relate to the patterns of retrieval latencies: subjects giving relational descriptions consistently had fastest latencies for cases where the symbols were the same, while subjects giving operational descriptions consistently had fastest latencies for cases where the second symbol was the yellow circle (identity element). This demonstrates that what is learnt is not simply a structure, but a particular encoding of that structure. This has implications when information about the structure has to be processed. For a subject using the relational encoding, it is easy to see that the structure is commutative; but it is difficult for a subject using the operational coding. The converse holds if the existence of an identity element is the question.

The experiment also implies that the subjects' verbal reports were valid reflections of their internal representations used in retrieval. A similar finding was obtained in an experiment using the set operation Complement of Difference (Mangan, 1981), which subjects describe in many different ways; the verbal descriptions were systematically related to objective performance measures, including retrieval latencies. This sort of finding is important in view of current controversy over the use of verbal reports as evidence (e.g. Evans, 1980; Ericsson and Simon, 1980). Certainly, as Kilpatrick (1981) pointed out, there is a need for verbal reports to be checked against results obtained by other methods, such as chronometric analysis.

Relevance to Mathematics Education?

The fact that mathematical structures underlie the experimental tasks we use does not mean that they have any direct implications for mathematics education, as Freudenthal (1973) has made clear. However, he does acknowledge that "this does not exclude that they may contribute in some way to the knowledge of the learning process" (p.503). We are investigating what Simon (1973) calls "a law-discovery process" which he defines as "a process for recoding, in parsimonious fashion, sets of empirical data". To the extent that this is a mathematical activity, a general psychological theory of such processes is relevant to our understanding of mathematical thinking.

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"Elementary Teachers' Knowledge of Mathematical Vocabulary"

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Purpose and Significance

The present study was an exploratory attempt to assess the vocabulary knowledge of inservice elementary school teachers of mathematics. It involved use of vocabulary commonly found in the elementary school mathematics curriculum.

Many educators have explored the relationship between mathematics and reading. These explorations have ranged from looking at one skill-area of reading such as comprehension to more comprehensive studies of the entire reading process as it relates to mathematics. The strong relationship between reading, language, and mathematical performance has been supported by a large number of research studies and writings by scholars in the field of mathematics education. The present study deals with the reading skill-area of vocabulary. It was assumed that if vocabulary is a crucial factor in a child's mathematical learning, then it is important to know how well classroom teachers deal with that same vocabulary. The present study was designed with that goal in mind.

Conceptual Framework

Successful reading in mathematics requires an understanding of the "two languages of mathematics": the technical vocabulary and the specialized symbols. Within these two languages of mathematics, there are five levels of response. They are: (1) letters, (2) words, (3) sentences, (4) paragraphs, and (5) discourse. For this hierarchical model, reading moves from small units to complete pieces of text or discourse.

The present study centered on level two - the word level. The potential problems in mathematical vocabulary fall into five broad categories. They are:

Words with More Than One Meaning - Many words which are familiar to most readers have special meanings in mathematics; for example, words such as square root, point, and slope. Square can mean "outdated," "a geometric figure" or "a mathematical operation (10^2)."

Words with Special Emphasis in Mathematics - Phrases such as "how many," "how many more," or "how many less" take on special meaning in mathematics. Failure to interpret them properly could lead to faulty solution of a mathematical problem.

Technical Vocabulary - Technical terminology in mathematics may present problems of three different kinds. First, the word may be entirely new. The student may be unable to pronounce the word or to use word-analysis skills. Second, the concept represented by the word may be new. Finally, the concept represented by a word may have no simple concrete referent. Some examples are sine, cosine, polynomial and chord.

Varied Forms - Another confusing factor in vocabulary development is that basic words can be presented in many different forms. The student will have to recognize differences in pronunciation as well as identify differencs in meaning. An example of this potential problem is found in the variations of the word multiply (multiplier, multiplication, and multiplicand).

Abbreviations and Specialized Symbols - A final area of potential difficulty is the use of abbreviations and special symbols. For example, cos for cosine and in, or" for inches. (note that" is also used for denoting seconds in degree-measure notations;" also has ordinary English usages.)

This categorization scheme was employed in devising the testing instrument for the present study.

Method

A group of 23 inservice elementary school teachers were tested during the summer of 1981. Each of the participants was presently employed in an elementary school teaching position as well as being currently enrolled in a graduate-level mathematics methods course at The University of Texas at Austin.

The test was a 50-item multiple-choice test on mathematics vocabulary. It was constructed by using 10 items for each of the five borad categories listed above. Words were selected by surveying several elementary school textbooks. Distractors were selected by looking at the most likely misinterpretations of the vocabulary chosen for the study. Each item had four (a, b, c, d) possible choices for the answer.

The tests were graded, taking note of the relative distribution of the selection of the various distractors. A descriptive analysis of the test results (maybe using percentages) would allow consideration of questions such as:

(1) Which type of vocabulary difficulty do teachers experience the most?
(2) When teachers misinterpret vocabulary, what is the nature of their errors?
(3) How does the teacher's knowledge as represented in (2) relate to children's perception of vocabulary as understood in the present literature?

Results and Conclusions

Teachers have a reasonable (?) knowledge of vocabulary in elementary school mathematics. Overall on the test, the subjects responded with an accuracy rate of 81%. By category, correct response percentages were as follows:

Words with more than one meaning:	83%
Phrases with special meaning:	82%
Technical Vocabulary	73%
Varied Forms	81%
Symbols	87%

This indicates that elementary teachers have the most trouble with technical vocabulary, which is not surprising.

If these results are confirmed over a series of studies it could suggest more emphasis on technical vocabulary in the mathematics education for teachers of elementary school mathematics. Further research will serve to clarify this topic.

NOTE 1: This study was a first attempt to explore the topic of teachers' knowledge of mathematical vocabulary. The study was done with a small sample ($n = 23$). Therefore, the results and conclusions to be drawn suffer from limitations. The researcher learned several things which will help to refine further investigations in this area.

NOTE 2: This abstract is limited in scope due to space limitations. The paper presentation will expand on all facets of the study. Handouts will be available at the presentation.

Some Suggested Readings

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THE SIGNAL VALUE OF WORDS IN MATHEMATICAL PHRASING

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Introduction

There are many words which, when used in mathematical texts, have a specific *signal value*. The word 'continuous', for instance, used in a mathematics book, will hardly ever be directly related to the natural language concept of continuity; on the contrary, the word brings about a reaction in the experienced reader such that he automatically chooses the 'right' (i.e. mathematical) meaning. Clearly, the signal value of the word 'continuous', when occurring in a mathematical text, decides on the concept that arises in the mind of the reader; for that reason we call it a *conceptual* signal. There are also certain words which emit a *contextual* signal by giving extra information on the environmental setting. For example, the use of the word 'suppose' in a mathematical text signals the start of an argument based on an assumption. Hence the word 'suppose' emits a signal concerning the context that can be expected: most probably the author intends to prove a universal statement or some form of an implication (cf. par. 2a of this paper).

Hereafter we shall discuss these two classes of mathematical signal words in more detail. Both classes will de subdivided, depending on the nature and the effects of these signals.

Right now we stress that the signal value of a word develops slowly in the mind, by repeated usage. For example, the word 'inequality' has a specific meaning in a mathematical context: it denotes an expression of the form $p \neq q$. However, in a learning situation, a student has to disengage himself from the notion 'inequality' as it already exists in his mind. In our opinion, this process consumes more time than teachers usually appreciate. In mathematical education one should be aware of this process and try to prevent conflicts in the development of understanding.

1. Conceptual signals

1a. *Natural language words*

In mathematical language one uses many natural language words with their normal meaning: 'example', 'the', 'if', 'use', 'with'. These words are 'signs' in the linguistic sense, according to their connotations (cf. the classification by Pierce (1931/1958)). However, we will not refer to these words as 'signals', because they do not evoke a special reaction in the reader, caused by the mathematical setting. In this respect we note that some natural language words are subject to a mathematical flavouring that causes them to drift away from their original meaning; in so doing they develop a signal value. For example, the word 'so' in mathematical phrasing has become, in the course of time, a logical-mathematical meta-connective; it signals a logically motivated conclusion, which is independent of any personal decision. Cf.: '$x > 1$, so $x > 0$', in contrast to 'I am tired, so I go home'.

A number of natural language words are given a new meaning in a mathematical context. In this way they acquire a mathematical signal value. This new meaning may agree with the usual one (e.g. 'maximum'), but there is often a considerable difference in meaning. Cf.: 'origin', 'complex', 'function', 'power', 'primitive'. This may give rise to conflicts. Some examples are: a limit of a sequence is not necessarily an (upper or lower) bound; with many mappings one cannot associate any depictable 'map'; one may have doubts about the 'reality' of real numbers.

1b. *Mathematical words*

In mathematics there is a standard notation for frequently used words: 0 for zero, Δ for triangle, etc. It is worth noting, however, that even standard notation can sometimes cause confusion. For example: $\mathbb{N}^2$ is not the set of all squares of natural numbers.

Some words occurring in mathematics are abbreviated directly from the original natural language word: lim for limit, ℓn for natural logarithm, etc. In usage, these abbreviations quickly detach themselves from their origins and they develop a signal value of their own. Standard abbreviations are not always unique. The symbol e, for example, can represent the base of the natural logarithm or the unit group element. One should not neglect the power of standard notation. When abbreviating 'length' by 'ℓn', one is asking for difficulties.

A number of mathematical words can be employed for various purposes; to this end one uses letters, possibly combined with numbers, subscripts, etc.: x, $a3$, $\ell\ell$, t_2, $M_{n,\ell}$, x'. Some of these words traditionally have restrictions in use: the letters i to n usually represent natural numbers or integers, x and y are real numbers, z a complex number, α an angle. Although there are no fixed rules for this usuage, one should respect these unwritten conventions because of the connected signal values. It is, for instance, didactically undesirable to define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(n) := x^n$. Here n is a real variable and x a constant. (Try to find the derivative of f!)

2. Contextual signals

2a. *Essential signals*

A number of contextual signals are *essential* for the course of the mathematical reasoning being presented. First of all we mention the *assumption* signals, which establish the hypothetical nature of a sentence in an argument. The word 'suppose' in the sentence 'Suppose g.c.d.(a,b) = 1' is such an assumption signal.

There are two kinds of assumptions, hence two kinds of assumption signals. Apart from the 'pure' assumption, as in the above sentence, there is a 'free' assumption, introducing a new name for an arbitrary object of some specific type, as is the case in 'Let x be a real number'. These two kinds of assumptions serve different purposes. The first one is used to prove an implication (e.g. 'If g.c.d.(a,b) = 1, then there exist integers x and y such that $ax + by = 1$'), the second serves to prove a generalization (e.g. 'For all real numbers x it holds that $x^2 - x + 1 > 0$').For a further discussion, see Donkers (1981). Since assumptions play an important role in arguments, it is regrettable that in common usage some assumption signals can have several possible interpretations. For example, the assumption signal 'assume' can be used for both kinds of assumptions. This fact diminishes its signal value. We note that this lack of distinctive power occurs with many contextual signals. We mention, for instance, the signal 'let...be', that serves as a definition signal in 'Let n be the smallest number for which $2^n > 1000$' (viz. n := 10), and not as an assumption signal.

Definition signals, too, are essential signals. Apart from 'let...be', we have, among others, 'take', 'is called', 'define...as', 'becomes', 'we indicate...with', 'consider ...to be'. In a sentence expressing 'x is defined by A', definition signals connect two parts, the definiens A and the definiendum x. .

Next, there are contextual signals expressing a logical connection between one part of an argument and another. These *connecting* signals occur in two versions: the *causal* signals and the *concluding* signals. Of the causal signals we mention: 'for', 'because', 'while', 'since', 'inasmuch as', 'namely', 'in consequence of', 'as a result of'. Examples of concluding signals are: 'thus', 'therefore', 'consequently', 'so that', 'hence', 'apparently', 'evidently', 'obviously' (cf. Anderson and Johnstone (1963), p. 60-61).

We note that concluding signals separate two parts of the reasoning which must necessarily occur in the natural deductive order: 'A, hence B'. With causal signals the order is mostly free: 'Since A, B' occurs next to 'B, since A'. (The signal 'for' always brings about the inverse order: 'B, for A'.) The logical connection caused by connecting signals can occur directly between two adjacent sentences ('$x > 1$, hence $x > 0$'), but a more indirect connection is possible as well.

2b. *Textual signals*

In mathematical texts one uses the same signals for denoting the textual structure as one does in ordinary phrasing. Although these signals are not special to mathematical texts, we will nevertheless mention them, since they often express the deductive structure implicitly present in a portion of text. For example, the signals 'on the one hand' and 'on the other hand', used in mathematical phrasing, will most probably signal a (logical) case analysis.

To begin with, we mention the *separation* signals. These signals interrupt the line of (directly connected) thoughts, and announce a new line. Some examples are: 'moreover', 'also', 'besides', 'likewise', 'now', 'on the contrary', 'but', 'however', 'yet', 'although'.

Next, there are *grouping* signals, dividing a text into connected parts. They often occur in pairs. Examples are: 'firstly', 'secondly', 'on the one hand', 'on the other hand', 'further', 'next', 'hereafter', 'finally', 'conversely'.

A *commenting* signal indicates that the author of a text is inserting a comment for the reader that stands apart from the actual reasoning. These commenting signals form an amorphous group, containing e.g. 'we shall prove', 'it suffices to show that', 'analogously', 'it is trivial that', 'apparently', 'it follows easily that', 'we recall'. The signal value of these word combinations is not very impressive.

A signal that is not expressed by a specific word, but by the mood used, occurs in proofs by *reductio ad absurdum*. Example: 'If n *were* a prime number, ...'.

Final remarks

The conventions and habits associated with mathematical signals have important consequences for the understanding of mathematical phrasing. Moreover, there are a considerable number of signals occurring in such texts. In our opinion, it would be interesting to know how a mathematical signal value evolves in the mind.

The development of conceptual signal values is connected to the building of *concept images* (see the description in Tall and Vinner (1981)). Contextual signals help to develop a *structure image* of an argument; this is an essential device for the understanding of complex logical arguments like proofs. As a side effect, knowledge concerning mathematical signals may be of aid in the production of didactically sound mathematics courses.

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MODES OF PRESENTATION FOR MATHEMATICAL STORY PROBLEMS*

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"A picture is worth a thousand words." Might this bit of folk wisdom apply to children's performance on the typical mathematical story problems? This question is the basis for the study reported here.

Background Mathematical story problems of a routine nature are a customary part of the curriculum for grades 3-7 (roughly ages 8-12 years) in the USA. Most often the problems are presented through words and numerals only, in a three-to-four line format (our *verbal* mode). Much less often in these grades are such problems presented through a *drawn* mode, with a drawing conveying the context. (The data and question are presented verbally.) Occasionally one also sees a verbal mode in an abbreviated form, a *telegraphic* mode, using sentence fragments rather than full sentences. Samples of each of these are in Figure 1.

The drawn and verbal modes have been contrasted in a study with 262 fifth-graders, with the drawn mode statistically superior (Threadgill-Sowder & Sowder, to appear). The study reported here is an extension of this earlier work to more grade levels, with an eye toward developmental trends. The telegraphic mode was introduced for two reasons: (1) We could find no research information on it, and (2) its presence could perhaps make clearer what features of the drawn mode matter—fewer words, or the presence of the drawing.

Why *should* a drawing help? Possible *a priori* reasons include these: (1) Drawings may be motivating; (2) drawings may lessen the reading load, a possible obstacle for some children; (3) drawings may provide images which some children cannot supply for themselves; and (4) drawings can organize the data. At least one of the reasons, reading load, is suspect,

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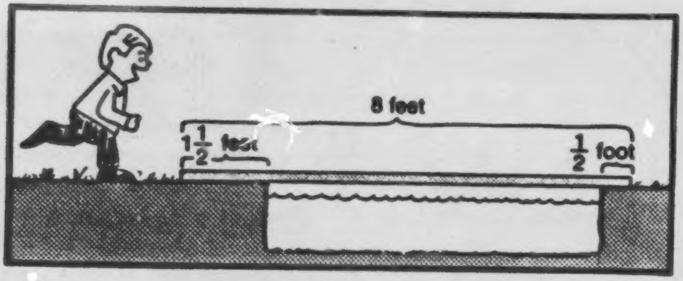
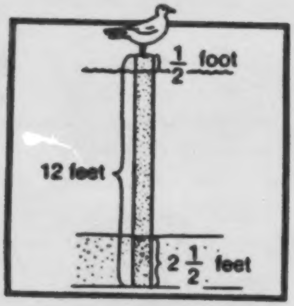


How wide is the stream?

A board 8 feet long is used to make a bridge over a stream. $1\frac{1}{2}$ feet of the board is on land on one side. $\frac{1}{2}$ foot is on land on the other side. How wide is the stream?

Stream
Board 8 feet long for bridge
$1\frac{1}{2}$ feet on land on one side
$\frac{1}{2}$ foot on land on other side
How wide is stream?



How deep is the river?

A post 12 feet long is pounded into the bottom of a river. $2\frac{1}{2}$ feet of the post are in the ground under the river. $\frac{1}{2}$ foot sticks out of the water. How deep is the river at that point?

Post 12 feet long
$2\frac{1}{2}$ feet in the ground under the river
$\frac{1}{2}$ foot sticks out of the water.
How deep is river?

Figure 1. Two companion problems in the three formats (grade 7).

since Knifong and Holtan (1977) analyzed children's work on story problems and concluded that most often, reading difficulties could not be blamed for their failures to solve problems. The other reasons can be defended. For example, Salomon (1974), as an explanation for the superiority of pictures to words in recognition tasks, posits that a picture "arouses more activity for the selection of cues, which are subsequently transformed, at least by adults, into one or more internal verbal descriptions" (p. 389). The organization aspect, reason 4, is no doubt a compelling reason for the oft-encouraged heuristic, make a drawing, for nonroutine problems.

It is important to note that the various reasons probably apply in different strengths for different children. This report concentrates on the overall group differences and represents only a part of our project. Threadgill-Sowder's presentation at this conference considers aspects of individual differences.

Development of the Problem Tests As Figure 1 suggests, each problem appears in the three formats and as part of a "family" of three "companions" (only two companions are in Figure 1). Companion problems have the same basic mathematical structure and involve comparable numbers but differ in some contextual features. A large-scale pilot of all the items generated gave evidence on the comparability of companions. From a family, a test for a given student would include three problems, one in each mode. Each test involved 8 families, so a complete Problem Test was made up of 8 verbal, 8 telegraphic, and 8 drawn problems. Multiple-step problems and problems with extraneous data were incorporated systematically. Three forms of each test (for a grade level) were designed so that each companion was represented in each mode. Each 24-item Problem Test was split into two parts, which were given in different orders. Half of the problems were repeated at the next grade level.

The number of operations involved in each test was 36, 12 for the 8 problems in each mode. Scoring was based on the number of correct choices of operation.

Subjects Children from three mid-continent cities were tested, 8 classrooms at each of grades 3-7. The classrooms represent 11 schools volunteered by administrators. The schools are located in differing socioeconomic areas.

Procedure The two parts of the Problem Test were administered in two sittings, each part included with batteries of cognitive tests (see "Individual Difference Variables and Story Problem Performance" for a description), Some schools had short class periods, so a third day was required to administer all the cognitive tests. Twenty to 25 minutes were allowed for each part of the Problem Test. Virtually all children finished in the time allowed.

Results Descriptive statistics are given in Table 1 for the tests scored at this time. A more complete analysis will be reported at the meeting. A glance at these incomplete results suggests that the drawn mode fares as well as the two verbal modes, although it may not give a statistically superior performance at all grade levels.

Grade	Mode	Verbal	Drawn	Telegraphic	Total
3	Mean	6.20	6.78	6.31	19.29
(n=103)	SD	3.50	3.37	3.63	9.87
4	Mean	6.69	7.65	6.71	21.05
(n=113)	SD	3.67	3.35	3.61	10.18
5	Mean	7.13	7.75	7.15	21.81
(n=110)	SD	3.51	3.07	3.42	9.55
6	Mean	6.45	6.61	6.14	19.19
(n=114)	SD	3.40	3.03	3.25	9.14
7	Mean	5.94	6.41	5.92	18.27
(n=122)	SD	2.89	2.94	3.13	8.19

Table 1. Means and standard deviations (SD) for each mode in the Problem Tests, by grade. Maximum score in each mode is 12.

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INDIVIDUAL DIFFERENCE VARIABLES AND STORY PROBLEM PERFORMANCE

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Background and Rationale

Most mathematics teachers would agree that no two students solve a mathematics problem in exactly the same way. It seems appropriate then, when researching student performance on problem solving, to consider individual difference variables likely to affect performance. In "Modes of Presentation for Mathematical Story Problems," also included in these proceedings, is a description of a study involving format variables used in presenting story problems. Another aspect of this study examines the effects of individual difference variables on problem solving. Age and sex are two obvious variables to consider. Another variable included (but not reported here) is the difference in performance by learning disabled and regular students. Finally, we included cognitive variables which have been shown to be related to problem solving performance. Perhaps the most obvious of these is reading ability, since the usual format of story problems is verbal in nature. The relationship between reading ability and problem solving performance has been well established (Aiken, 1972).

The second cognitive variable we chose to study was spatial ability, long thought to be related to mathematical ability (Smith, 1964; Bishop, 1980). More specifically, research has often suggested that spatial ability relates positively to problem solving performance (e.g., Fennema & Sherman, 1977). An attractive hypothesis in the present study is that a visual format for presenting problems facilitates representation and processing by individuals. For example, a learner who does not easily generate a mental picture

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from a word description might benefit from a drawn version of a story problem. The drawn version would serve as what Cronbach and Snow call a "prosthesis" (1977, p. 282). Although we realize that the ability to form visual images is not the same as spatial visualization ability, we believe that spatial visualization measures might aid in locating good problem solvers, particularly those who do not need assistance from drawings.

Problem solving ability has also been found to be related to field dependence-independence. This construct can be conceptualized as an articulated-global continuum encompassing a wide range of perceptual and intellectual situations. It has been found to be related to problem solving ability (e.g., Bien, 1975; Ehr, 1980). In particular, an interaction has been found between FDI and treatments involving verbal and drawn formats for story problems (Threadgill-Sowder & Sowder, to appear). It may be hypothesized that for the field dependent learner, a drawn version of a problem might result in greater prominence of the essential information than does a verbal version.

Measures

Two recommendations by Cronbach and Snow (1977) influenced the number of measures selected. (1) Even when investigators are chiefly concerned about one or more special abilities, they should include at least one general measure to test the hypothesis that general ability explains the aptitude-treatment (or in our case, aptitude-format) interaction. Test 2 (Classifications) from Scale 2, Form A, of Test of "g": Culture Fair from the Institute for Personality and Ability Testing was selected as a general measure of fluid intelligence (abbreviated as CFT here).

(2) The second recommendation was that more than one test be used to measure special abilities. Two tests were selected to measure spatial visualization ability. One is the Spatial Relations Test (SRT) from the Primary Mental Abilities tests, grades 4-6, published by Science Research Associates. The second spatial test is the Punched Holes Test (PHT), adapted from the Kit of Reference Tests for Cognitive Factors published by

Educational Testing Service. The two tests selected to measure field dependence-independence were the Hidden Figures Test (HFT) and the Find A Shape Puzzle (FASP). The HFT is a simplified version of a test from the Kit of Reference Tests for Cognitive Factors, and the FASP is from Pulos and Linn at Lawarence Hall of Science, Berkeley. Because of the length of most reading tests, only one test was selected to measure reading, the Test of Reading Comprehension (TORC), Subtest #2—Syntactic Similarities, published by Pro-Ed of Austin, Texas. Finally, the Problem Solving Tests are those described in "Modes of Presentation..." elsewhere in this volume. Only the problem tests differed with grade level.

Procedure

All tests were administered to students in 40 classrooms, eight in each of grades 3, 4, 5, 6, and 7. Eleven different schools located in three cities and representing different socioeconomic areas participated. The testing is more fully described in the "Modes of Presentation..." paper.

Results

Means and standard deviations on the cognitive tests are shown in Table 1. Correlations are shown in Table 2.

TABLE 1.

MEANS AND STANDARD DEVIATIONS ON COGNITIVE TESTS

Grade Level		Culture Fair Test Max=14	Hidden Figures Test Max=12	Spatial Relations Test Max=25	Punched Holes Test Max=10	Test of Reading Comp. Max=20	Find A Shape Puzzle Max=20
3	M	6.26	4.21	12.59	3.62	7.80	4.65
n=103	s	1.73	2.27	4.00	1.66	4.00	3.30
4	M	6.57	5.02	14.36	4.48	9.05	6.84
n=113	s	1.74	2.20	4.36	1.78	3.94	4.21
5	M	7.14	5.88	15.05	5.22	10.58	7.99
n=110	s	1.58	2.70	3.84	1.90	4.38	4.71
6	M	7.40	6.50	16.48	5.54	11.97	9.95
n=114	s	1.66	2.88	4.13	2.23	4.61	5.28
7	M	7.23	6.68	16.99	5.81	12.05	9.55
n=122	s	1.58	2.60	3.76	2.06	4.32	4.64

TABLE 2

CORRELATIONS[1] BETWEEN TESTS AT EACH GRADE LEVEL

	Grade[2]	HFT	SRT	PHT	TORC	FASP	PST[3]
Culture Fair Test	3	.06	.39	.08	.27	.23	.15
	4	.06	.13	.06	.14	.10	.08
	5	.04	.25	.35	.30	.04	.35
	6	.37	.36	.36	.38	.45	.43
	7	.04	.16	.24	.28	.16	.14
Hidden Figures Test	3		.30	.23	.16	.37	.24
	4		.14	.39	.21	.57	.28
	5		.40	.38	.22	.52	.31
	6		.45	.43	.41	.59	.48
	7		.41	.32	.21	.56	.39
Spatial Relations Test	3			.42	.31	.52	.39
	4			.32	.21	.42	.17
	5			.41	.35	.45	.46
	6			.56	.37	.63	.46
	7			.42	.20	.58	.46
Punched Holes Test	3				.23	.34	.28
	4				.31	.45	.29
	5				.47	.49	.36
	6				.43	.58	.53
	7				.37	.36	.44
Test of Reading Comprehension	3					.27	.55
	4					.44	.47
	5					.44	.63
	6					.63	.68
	7					.32	.55
Find A Shape Puzzle	3						.34
	4						.33
	5						.42
	6						.63
	7						.47

Notes:
1. Significant at .05 level if <.15, significant at .01 level if <.21 (approximate values)
2. n=103 at Grade 3, 113 at Grade 4, 110 at Grade 5, 122 at Grade 6, 128 at Grade 7.
3. PST is Problem Solving Test, Total Score from all three subtests.

As one would expect, scores generally increase at each grade level. Correlations between HFT and FASP (the two tests for field dependence-independence) are reasonably high, but correlations between SRT and PHT (the two tests of spatial ability) are not even as high as those between SRT and FASP. TORC (the reading test) seems to be a better predictor of problem solving ability than the other tests. (Data in these tables are based on students in 5 classrooms at each of grades 3-6, and 6 classrooms at grade 7.)

T-tests between scores for males and scores for females on each test indicated that boys do better on field dependence-independence tests, spatial tests, and problem tests, and that girls received higher scores on the reading test. While these findings were consistent at each grade level, the only t-tests which were significant ($p < .05$) were TORC at grade 4, SRT at grades 5 and 6, and HFT and PHT at grade 6.

Further analyses on the data will take place in the next two months and will be reported on during the conference.

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D. PROOF AND VERIFICATION OF TRUTH

DIFFICULTIES AND ERRORS IN GEOMETRIC PROOFS BY GRADE 7 PUPILS

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Among the subject areas in secondary school mathematics geometry is the one in which proving is mostly required and which therefore is regarded as the most suitable domain to train pupils in proving. By the term "proving" we sum up a wide field of activities, such as assessment of given proofs with respect to consistency, reproducing learned proofs as well as inventing a proof, that is constructing a line of reasoning for given statements, the parts of it being more or less known. These abilities are so difficult to be attained, that they only can be trained in a long-term process. Therefore, analysing and identifying difficulties and errors in geometric proofs may be an important remedy to learn proving.

Error analyses, as far as they refer to a specific subject area of mathematics, are to a high extent confined to arithmetic. In this field many outcomes are known. In addition to this we have investigations into achieving algebraic techniques. Error analyses,with respect to geometric topics, are restricted to the acquisition of fundamental concepts. Proving in geometry is a domain in which a lot of work has to be done.

A wide field of research into errors has been carried out, which is not content-specific, and which is often based upon special psychological approaches, such as psycho-analysis, Gestalt theory.

With good reasons, geometry is regarded as subject area in which systematic investigation into errors is difficult. Not only the formation of concepts and their correct use is crucial for the successful treatment of tasks and problems, but above all the complex ability to form problem solving strategies and to apply them in a correct and adequate way.

Especially in analysing abilities of proving, such explanatory models for thinking processes, which are based on the analogy with information processing, seem to be suitable. This is due to the fact that a proof aims at exploiting the premise in such a way as to make obvious, that it contains, logically, the conclusion, too.

In terms of information processing, a problem is characterised by three components: 1. a starting point to be subdued, 2. a final state to be attained, 3. a barrier which prevents the transformation of the starting state into the final state at the beginning (cf. Dörner, 1979). Transformations are done by operators, the successive application of a sequence of operators links the starting state with the final state by a chain of intermediate states. Any specific domain is characterised by its possible states as well as by its operators, acting on the states.

To solve a problem, it is necessary, beyond reactivating suitable knowledge, to combine known transformations in an adequate way, or even to construct still unknown transformations. Thus, the problem-solver has to dispose of construction procedures called heurisms; these are plans which are fixed more or less precisely, to enable him to produce transformations.

Furthermore, we have to consider, that different forms of memory take over discernable functions during a problem solving process; storage capacity and residence time characterise in different ways sensory memory, short-term memory and long-term memory.

If we apply this general approach to the special problem type to give a proof of a geometric theorem, we can say that states are the theorems which hold for the geometric figure in question, more precisely: lists of statements holding for the figure or parts of it, which are accepted as granted in the respective stage of the procedure. Such a list would contain, at the beginning of the proof, the specific premise, in case of a triangle for instance furthermore the angle sum theorem for triangles, and so on. Because of the small storage capacity of the short-term memory the problem-solver concentrates above all on the premise. Any operator amplifies this list to a larger one, which contains, as further statements, inferences from the already present statements; some of the inferences claim special interest, attention is focused upon these, as they are regarded as generating the steps by which the goal can be reached.

Strategies which generate conditions for applying operators to already reached states, such as the strategy to complete the starting figure by an auxiliary line or a supplementary figure so that figures with known properties may emerge, are heurisms, as well as selecting processes, such as selecting the proof form (indirect proof, mathematical induction) (cf. Witsel, 198o), checking processes, and even superior strategies to rule subordinate strategies.

An important component of proving, as opposed to steps in information processing, is the presentation of proofs in language medium. Proving is a "public activity" (Bell, 1976), which means that each proof step must be communicable. Encoding and decoding, therefore, constitute an essential component, and, at the same time, a possible source of error.

Thus, we have the conception, that every proof error results from 4 components: 1. the phases referred to in the sketched model, 2. the language component, 3. situative subjective aspects, and 4. situative objective aspects. To the situative subjective aspects belong short-term operating conditions such as motivation, weariness, effects of Einstellung, and also effects of permanent personality variables. To the situative objective aspects belong the problem-solver's knowledge, especially his acquaintance with special cases, certain rules, the special properties of the proof figure, etc. It seems to be unnecessary to emphasize that the four components influence each other mutually and form complex bundles of causes for error and deficiencies in proving.

The subsequent classification is oriented to the above mentioned progress model to generate a problem solution. It shows common traits with a classification given by Raddatz (Raddatz, 198o), which is also based on information processing, but which claims not to be linked to any specific subject field within mathematics.

Most of the examples to be given in the presentation are taken from written proof tasks and systematically controlled homeworks by grade 7 pupils, and partly from classwork. Topics for mathematics instruction were some elementary geometric theorems, the emphasis being placed on proving and the acquisition of basic proving skills.

The following error categories can be distinguished:

1. Deficit in specific knowledge

That might be that knowledge needed is not available or not accessible in memory (cf. Tulving/ Pearlstone, 1966). And even if topics to be linked together can be retrieved, the problem-solver has to use certain heuristic strategies, which might be unknown or inaccessible.

2. Trial-and-error-strategies

In àll classifications of failures and deficiencies we can find the category of random responses or haphazard strategy. Pupils often prefer to give any answer - whether correct or not - than to give no answer.

3. Errors by inadequate understanding of single elements,

such as technical terms, symbols, parts of a figure or a text, and so on.

4. Errors by inadequate understanding of higher units,

such as theorems as a whole, grammatical structure of sentences, complex logical connections between parts of a theorem or a series of several theorems.

5. Errors caused by changing information when encoding, decoding or recoding

Those errors can be observed especially when a person is obliged to express a solution, a thought, or parts of reasoning, in language.

6. Errors caused by prevalence of single parts of a sentence, of a figure, or of certain parts of a configuration of elements

Centration upon the "most promising" parts of the search space is a successful strategy in many cases, but on the other hand fixation on certain parts, perseveration, and effects of Einstellung may cause errors and prevent the problem-solver from finding a solution.

7. Errors caused by applying an operator inadequately

f.i. non-separating similar operators (a special case of reacting to similar stimuli), lack of checking whether or not an operator is applicable.

8. Loss of information when applying an operator

The problem-solver tries again to prove certain properties of a figure, which have been known before, f.i. premises.

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Proof and Certitude in the Development of Mathematical Thinking

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The question which inspired the present research was the following: Does the high-school student, normally involved in courses of mathematics, physics, etc., clearly understand that a formal proof of a mathematical statement confers on it the attribute of a priori, universal validity - and thus excludes the need for any further checks? Or will he assume that more checks are always desirable in order to evaluate the validity of the theorem (already proved)?

The main hypothesis of the present research was that most of the students - even those attending mathematical classes - do not have a clear idea of what a formal mathematical proof means. We have assumed that these students, after finding or learning a correct proof for a certain mathematical statement will continue to consider that surprises are still possible, that further checks are desirable in order to render the respective statement more trustworthy (as it happens in empirical research).

The rationale of that hypothesis is the following. From a common sense, and, in general, from an empirical standpoint, the Kantian concept of synthetical judgements a priori has no meaning. For the current understanding, a statement can either by self-evident - and thus it is intrinsically universal (any proof appears as being a superfluous act); or not directly obvious, i.e. its acceptance requires a proof - and thus belongs to the category of experimental findings: its degrees of generality and validity depend on the quantum of confirmations. Thus, for common sense, for the empirical, ordinary approach, a certain statement can be accepted as valide either because it is intuitively evident or because it is the generalisation of reproducible findings.

As we said, we have assumed that the adolescent will encounter great difficulties in understanding and effectively using the concept of formal, universal validity and that his currently empirical ways of thinking - deeply rooted in his practical behaviour - will, generally, distort his decisions with regard to the certitude and validity of mathematical statements.

METHOD

In order to neutralise, as much as possible, the role of information and the information processing effect, the students have been provided with all the data needed for understanding the mathematical statements used in our questionnaires and the respective formal proofs.

The subjects were 397 students enrolled in three schools, located in different districts of Tel Aviv. The subjects belonged to the following types of classes: mathematics, electronics, mechanics, biology, humanities, foreign languages. According to the mathematical curricula used in these classes, the subjects were classified into two categories: a) those receiving a high mathematical training (HMT): the classes of mathematics, electronics and biology (150 subjects); and b) pupils with a low mathematical training (LMT): the classes of humanities and mechanics (247 subjects). Clearly, the two groups were not homogenous but what was important for us was especially to identify those pupils who received a high mathematical training in contrast to those who, according to the teaching programs, received only an elementary mathematical training. Three grade levels were investigated: 10, 11 and 12 corresponding roughly to the ages 15, 16 and 17. In total, sixteen classes participated in the research: six of grade 10, six of grade 11 and four of grade 12.

The Questionnaires

Two questionnaires were used, one with algebraic content and the other with a geometric content. The questionnaires were administered simultaneously, that is, part of the subjects received the algebraic questionnaire (N=200) and the others received the geometry questionnaire (N=197) during the same session. The questionnaires were administrated randomly in the natural classroom environment of the subjects.

Each of the two questionnaires consisted of A) an "Introduction", B) the "Core of the questionnaire" divided into two parts: B1 and B2, and C) a final, general question.

A. In the "Introduction" there were questions asking the subjects to perform some mathematical operations (calculations, or geometrical constructions) and to try to generalise and explain the results obtained.

B1. A theorem was given which related to the previous findings. The theorem was followed by the respective proof of it. The subjects were asked if they consider the proof to be correct and if, conseqently, they accepted the general validity of it.

The following theorems were referred to:

1) ABCD is a quadrilateral and P, Q, R, S are the midpoints of its sides.

One must prove that PQRS is a parallelogram.

2) The expression $E = n^3 - n$ is divisible for every n, n being any positive integer.

B2. This part was designed to determine if the subjects are consistent in their attitude, expressed in Part B1. In other words, will a subject, who has already declared his complete agreement with the theorem and its proof, totally refute any utility of further empirical checkings?

C. Final question: In the last part of the questionnaire - the subjects are confronted with a theoretical question. The subjects are asked to compare, theoretically, the modes of proof in mathematics and in physical sciences.

Examples of questions asked in Part B2 of the questionnaires:

Geometry: "V is a doubter. He thinks that we have to check at least a hundred quadrilaterals in order to be sure that PQRS is a parallelogram. What is your opinion? Explain your answer."

Algebra: "M claims that he checked for n=2357 and that he obtained: $n^3 - n = 2357^3 - 2357 = 105.513.223$ and this result is not divisible by 6. What is your opinion?"

"If you consider that Dan has given a correct proof for the theorem "$n^3 - n$ is divisible by 6 for every n" then answer the following question: Do you consider that further checks (by using other numbers) are necessary in order to increase your confidence in the validity of the theorem? Yes/No. Explain your answer."

The Results

We have considered as especially relevant the following patterns of answers, expressing the cognitive attitude and the consistency with regard to the mathematical (formal) mode of proof.

1) Consistently formal: Those subjects who a) in part B1 of the questionnaire agree with the general validity of the statement, agree with the correctness of the proof, agree that the statement is completely supported by the proof and that b) in part B2 of the questionnaire are able to defend their theoretical attitudes by correctly rejecting every (illegitimate) limitation of the generality of the proven statement and any need for further, empirical checks.

2) Basically inconsistent: Those who a) appear as being consistently formal in part B1 (agree with the theorem, with the proof etc.) and, despite this, manifest an empirical attitude in part B2 (the theorem may not be confirmed in some particular situations, further checks are considered necessary or, at least, useful, etc.).

3) Consistently empirical: Those subjects who a) in B1 do not agree with the principle that a formal proof can guarantee the general validity of a statement and b) continue to be consistent with that opinion in B2.

4) Reject the proof: A fourth category consists of those who simply do not agree with the proof and consequently cannot be classified in one of these categories for that reason.

5) Others: Finally, there are the rest of the subjects, those who gave irrelevant answers, who were generally inconsistent, who did not answer various questions.

In fact there are three basic relevant patterns of cognitive attitudes:
a) consistently formal; b) basically inconsistent; c) consistently empirical.

An overall presentation of the results for geometry and for algebra is given in Table 1.

Table 1

Percentages of Cognitive Patterns

		Geometry (N=73)	Algebra (N=77)
HMT classes	1. Consistently formal	11.0	23.4
	2. Basically inconsistent	34.3	25.9
	Consistent in B1 (1+2)	45.3	49.3
	3. Consistently empirical	2.7	15.6
	4. Reject the proof	2.7	6.5
	5. Others	49.3	28.6

Table 1 cont'd

		Geometry (N=124)	Algebra (N=77)
LMT classes	1. Consistently formal	6.4	8.9
	2. Basically inconsistent	32.3	28.5
	Consistent in B1 (1+2)	38.7	37.4
	3. Consistently empirical	7.2	30.9
	4. Reject the proof	8.1	12.2
	5. Others	46.0	19.5

The results confirm the basic hypothesis of our research: most of the subjects favour supplementary checks of an already proven mathematical statement, even if they have previously expressed their full agreement with the statement and its formal proof. This conclusion holds not only for students with a low level of mathematical education but also for students who, according to the teaching programs, were supposed to have a strong background in mathematics. We have found that, out of a total of 396 students, about forty per cent explicitly agreed with the mathematical statements presented and the respective proofs, but only 10.8% for geometry and only 14.5% for algebra were consistent in rejecting any need for further empirical checks. Considering only those subjects who expressed their agreement with the statements and the proofs (N=165), we found that 36% of the students with high mathematical training (HMT) and 20% of those with low mathematical training (LMT) understood that further checks are superfluous since a formal proof guarantees a priori the absolute validity of the statement.

Our basic explanation of this finding is that the ordinary student is intuitively inclined towards an empirical interpretation of the validity of an argumentation. He naturally tends to seek support for a statement by accumulating as many as possible significant confirmations. Of course, the student is unaware of that bias. In his overt, "surface structure" reactions he may clearly distinguish between formal and empirical proofs. But when the confidence of the student in the predictive validity of a statement which has been proved formally, is confronted with some non-standard questions, it may be found that his "deep structure" attitudes still remain anchored in his behavioural tendency to look for more and more confirmations.

HOW COMMUNICATIVE IS A PROOF?

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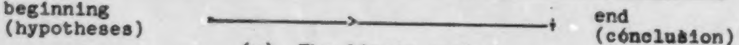
Mathematical proofs are normally presented in a step-by-step, "linear" fashion, proceeding unidirectionally from hypotheses to conclusion. While this age-old and venerable method may be well suited for securing the validity of proofs, it is nonetheless unsuitable for a second, highly important role of most presentations -- that of mathematical communication.

In this presentation an alternative method, called the "structural method", is proposed. The method, triggered by recent ideas from computer science, is intended to increase the comprehensibility of mathematical presentations while retaining their rigor. The basic idea underlying the structural method is to arrange the proof in levels, proceeding from the top down; the levels themselves consist of short autonomous "modules", each embodying one major idea of the proof.

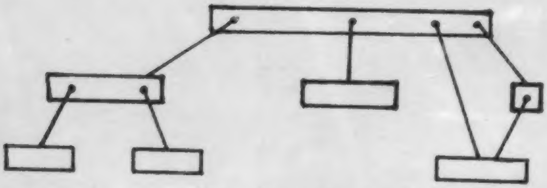
The top level gives in very general (but precise) terms the main line of the proof. The second level elaborates on the generalities of the top level, supplying proofs for unsubstantiated statements, details for general descriptions, specific constructions for objects whose existence has been merely asserted, and so on. If some such subprocedure is itself complicated, we may choose to give it in the second level only a "top-level description", pushing the details further down to lower levels. And so we continue down the hierarchy of sub-procedures, each supplying more details to plug in holes at higher levels, until we reach the bottom where (to borrow W. W. Sawyer's metaphor) all the leaks are plugged and the proof is watertight.

The top level is normally very short and free of technical (i.e. notational, computational, etc.) details. Thus it can be grasped at one glance, yielding an overview of the proof. (Note that the very term "overview" suggests view from the top.) The bottom level is quite detailed, resembling in this respect the standard linear proof. However, these details now appear only after their role in the proof is determined. Furthermore, they are now organized into conceptual units (the modules), each clearly and explicitly connected to its appropriate place in the total hierarchy. The intermediate levels facilitate a smooth transition from the generalities of the top level to the details of the bottom, from the global to the local perspective.

The two approaches are compared pictorially in Fig. 1. The linear method is represented by an oriented line segment (a), the structural method by a "structure diagram" (b). The structure diagram displays the levels, the modules and their interconnections. In each box, or module, the argument flows linearly, but it is very short and "flat" (no complex nesting



(a) The linear method



(b) The structural method

Fig. 1. The two methods of presentation

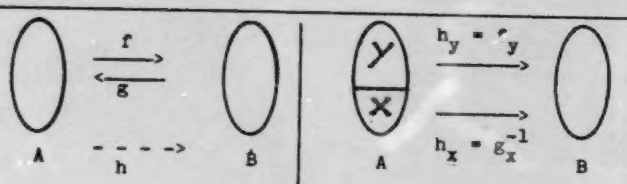
patterns of sublevels); thus, again, it can be grasped at a glance. Incidentally, the above description applies not only to proofs, but to other mathematical procedures such as definitions, constructions, algorithms and examples as well.

It should be stressed that many teachers and expositors do use occasionally structural ideas in their presentations. However, there are at least as many who adhere to the official linear style, and in fact consider any deviation from it "illegal" or unrigorous. It is therefore hoped that this paper will help provide a coherent and explicit system of presentation, whereby some of the "options" of good teachers will have become standard. Note that these options (e.g. a short overview of a long and complicated proof) not only become standard; they actually become part of the "official", formal proof.

It should also be stressed that the proposed modification bears directly only on one part of the teaching-learning process: the formal presentation of mathematical procedures, as they commonly occur in classes, textbooks and journal articles. By no means is it meant to replace the informal and artful devices that good expositors have always been employing to enhance the full and active participation of the student in this process, such as intuition, heuristics, personal metaphors, humor and even acting. After all, if the student is not listening, it matters little what we are saying!

By way of demonstration, let us consider briefly the Cantor-Bernstein Theorem of Set Theory:

If A and B are sets, and if there exist one-to-one maps from A to B and from B to A, then there exists a one-to-one map from A onto B. Thus, given two one-to-one maps $f : A \to B$ and $g : B \to A$, we wish to construct a one-to-one "onto" map $h : A \to B$ (Fig. 2 (a)).



(a) The theorem

(b) The top level of the proof

Fig. 2: The Cantor-Bernstein Theorem

One of the shortest proofs of this theorem when presented linearly, starts as follows:

Let $X_0 = g(B)$, $Y_0 = A - X_0$, $\varphi = g \circ f$, and define $Y = Y_0 \cup \left(\bigcup_{n=1}^{\infty} \varphi^n (Y_0) \right)$ and $X = A - Y$.

Note that $Y \supseteq Y_0$ so $X \subseteq X_0 = g(B)$. Thus $g^{-1}(a)$ is defined for $a \in X$ (since g is one-to-one),etc...

Compare this opening with the top level that starts the structured proof:

Level 1: Given a partition $A = X \cup Y$, one may try to define the function $h : A \to B$ by:

$$h(a) = \begin{cases} f(a) & \text{if } a \in Y \\ g^{-1}(a) & \text{if } a \in X. \end{cases}$$

(See Fig. 2(b).) We shall find (in Level 2) such a partition that will make h well-defined, one-to-one and "onto". This will complete the proof of the theorem.

The complete structured proof has 4 levels, leading gradually and naturally to the explicit definition of X and Y at the bottom.

More detailed comparisons of proofs in the two styles will be given in the Conference presentation.

Is '2+8=10' a Primitive or a Compound Proposition?

Abraham Shammas*

Abstract. The truth value of statements in mathematics plays an essential role. Correctness of algorithms or their implementations seem tightly related to the truth of the statements about their results. Correctness of performance in general seems to have psychological significance. With that, little is known about the cognitive components of either truth or correctness. Compound statements may be verified through resorting to the truth values of their primitive components. Not so with the primitives. Here, direct verification with the reality seems essential. What can be said about learning arithmetic? Is the statement'2+8=10' a primitive one?

Key Words: Model of mind, Mathematics Education, Cognitive Psychology, Truth, Semantics, Linguistic,Concept Formation, Philosophy of Mathematics.

A. Introduction.

Mathematics concentrates on true statements. These statements are usually compound propositions which include logical connectives (and, or, if, not, iff, etc.), or quantifiers (all, some). Theorems are usually about collections of objects using (implicitly) quantifiers, and/or connectives. And, they are verified through proof, using other statements. A special case may arise when we teach arithmetic. Mathematically, there is only one object 2, there is a unique object 8, and a unique result 10. Moreover, in formal linguistic analysis, the statement;

(1) '2+8=10'

or its equivalent in the ordinary language;

(2) 'Two plus eight is equal to ten,'

does not include any connectives or quantifiers. Thus, it seems fair to perceive it as a primitive proposition about the unique objects 2, 8, and 10. We enclose these statements in a pair of " ' " marks to distinguish the linguistic from the semantic aspects; The sign (1) stands for the mathematical statement

(3) 2+8=10.

Thus, (1) is true iff (3). But is it really a primitive? As we will shortly see, it actually depends on how the concepts NUMBER and ADDITION are being defined. Under some interpretations, the sentence (1) behaves like a compound statement. Each such interpretations has educational utilities (see Nesher, for example).

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B. '2+8=10' as a Primitive Statement.

In Cognitive Science, where computers are used to simulate the cognitive components of language processing, frames are being used to also model some aspects of mathematics development (see Davis, 1981; and Schank,1975). According to this approach, there are frames with slots for constants to come. For addition, for example, one can imagine a basic frame A(x,y,z), where the signs x,y stand for the constants to be added, and z for the result. Whenever the subject faces a specific case, say 2+5=7, a copy of A(x,y,z), saturated with the constants 2, 5, and 7 is established and linked to the basic frame A. When another new specific frame, say 3+5=8, is stablished, its link to the basic frame may be activated and this, in turn, may activate the older specific frames, say that of 2+5=7, enabling the subject to recall the latter. In this manner, the subject does not only recall other specific cases of addition, but also has direct access to their pattern; the adding super frame A(x,y,z). This approach to addition, mathematically, is legitimate. This is consistent with the relational approach to addition: a+b=c is perceived as a tri-variable relation, +(a,b;c). Accordingly, each specific case like (3) is an instance , and can be learnt through (now psychologically) an instantiation of the frame A, by inserting the constants in its slots. Then (3) is (mathematically) an instance of +(2,8;10), and (psychologically) an instance of A(2,8,10), and these are similar. The truth of (1) depends on the actual relation (3) between the objects named. Thus, each case like (3), can be seen as an individual episode whose truth should be verified through the analysis of the constants directly involved (e.g. +(2,3;5) is true, while +(2,3;4) is false). Thus the frame model, at least theoretically is capable of adopting the view that '2+8=10' is a primitive statement. Here, the objects involved seem to behave rather independently from each other, forced to have some relationships under specific conditions (or circumstances). One might have a better feeling of such circumstantial relations between the numbers in the following example: 2+8 is 0,1,2,3,4, or 10, depending on the modul (10,9,8,7,6, or infinity, respectively) you choose.

The problem with such approach is that some essential properties of number would be very hard to model. We would need independent basic frames for primes, base (change), processes of addition and so forth. It is not clear how the system would make inference from one frame to the rest.

C. '2+8=10' as a Compound.

In contrast to said above, other definitions of number and adding process may indicate that despite its rather simple form, the statement '2+8=10' behaves like a compound sentence. First, according to Peano, 2+8 is the successor of 2+7, AND that is the successor of 2+6, AND so forth. In practice such definition of concept number may be evidenced by the common practice of counting, from 2 up to the result 10, where the eight units of the addendum are totally consumed. So, for a young student, so it seems, the justification of $2+8=10$ (or that the statement (1) is true) rests on realizing that:

(4) $2+1 = 3$ AND $2+2 = (2+1)+1 = 3+1 = 4$ AND ... AND $2+8 = (2+7)+1 = 9+1 = 10$.

This, for a school child, seems more than just discovering the fact that $2+8=10$, as an isolated incident. At this stage, at least psychologically, (1) seems equivalent to the compound statement (4). But there is more to come.
We get a different definition of number, when we try to use set theory. According to set theoretical approach, $2+8=10$ is an abstract generalization over very many cases where the union of a set, with two objects, with another set, of eight elements, is being produced. Then (3) seems equivalent to :

(5) Two apples plus eight oranges equal ten fruits AND
two boys together with eight girls are ten children AND
two birds and eight (other) birds are together 10 birds AND
.
.
.

(For a review of some different definitions of concept number and references to curriculum using them, see Nesher 1972).
Then, again '$2+8=10$' is a compound sentence, composed of infinite number of primitives. And, there is still another sense in which '$2+8=10$' seems as compound, but let us delay its discussion for a short moment. Meanwhile it seems worth mentioning that such interpretation of concept number has found several psychological models. The best known model is the structural approach to concepts, introduced by Skemp(1979,1981). According to this view, the concept 2, for example, may be abstracted from some examples (abstract though they themselves may be), 8 from other examples, and 10 from still other examples. Then, presumably, the objective contingencies between the objects and their collections, force the construction of a schema for each concept, as well as their interaction: the schema for addition. 'Two boys together with eight girls' is an example of addition, '10 children' is an example of 10, and these are related through the schema of 'equality'. The schemas grow more and more, modify each other, and build up more and more accurate schemas. This seems to be a perfect counterpart to set theoretical approach to the concept number. Thus in this model, '$2+8=10$', for early stages of mathematical development (and possibly beyond), is a compound sentence like (5). Its truth should be established and verified through analysis of its compound structure. Its structure is a conjunction of primitives, each being an isolated case whose truth is established through direct investigation of, and/

or experience with reality.

Another model for the conceptual approach to numbers is HEIKR introduced by Shammas (1981, a & b). It is an image-based, rather than schematic model. Images of the domain (including, say, two objects) are accumulated in snapshots (over time), analyzed for their common attributes, intensionalized to produce more abstract attributes, and later, used as filters and supporting material in further mapping of the domain. Intensions (common attributes- after Woods, 1975), coupled with signs for objects, produce concepts, which can be used to produce more abstract concepts. Then, the statement '2+3=5' is a sign for an abstract concept (relation type) which can be supported by all images involved in its build-up. Its truth can be checked by resorting to abstractions of abstractions of...abstractions of real objects involved in past experience. In order for the sentence to be true (for the subject), it is necessary that all the supporting images and relations are CORRECT.

So we see that both models mentioned here are close to the set theoretical approach rather than the Peano type definition. The latter is more a procedural type. And now we come to a third sense in which (3) can be seen as a compound statement.

Let us concentrate, first on the clause

(6) '2+8'.

The string in (6) is not only a name for the final product 10. Moreso, it is a name for processes which combine the objects 2 and 8 into another object, 10 (rather than 16 which is their product under "."). But the clause (6) is free of any connotation about the specific adding procedure applied. Moreover, each such adding procedure and its instantiation could be checked for their CORRECTNESS. There seems to be an isomorphism between the correctness of algorithms used (or the specific performance of such algorithm), and the truth of the statement about the result. Now, for small numbers, such as 2 and 8 in '2+8', the correctness of the processes may merge with the truth value we attach to the statement about the experiences building our schema. Not so with large numbers: When I state

(7) 3590423+8919708=12520131,

I barely have any examples for such numbers based on "real experience". Nor it seems that I pay any attention to the basic adding frame mentioned above. Rather, the process of adding these large numbers takes over. The truth value of (7) is checked and rechecked, not through resoting to set examples, but through CORRECT application, of APPROPRIATE algorihms for addition. Then we tend to conclude that 'a+b=c' is true if and only if the following (over all processes P_i of addition) is true:

(8) P_1(a,b) gives c AND P_2(a,b) AND P_3(a,b) gives c AND...

This would mean that a procedural definition of addition would entail the conclusion that (1) is compound. This is a matter of fact that there are different algorithms for addition. Computers and calculators use different algorithms as compared with human beings. This will be interesting to further investigate the process of decision making when different processes give conflicting results. Mitnick(1982) reports about evidence that, while working with calculators, even adults prefer its (faulty) results (the students were not aware that the calculator they were using had in-built error), on their own (good) estimates. If that is true (as it seems judging from everyday experience with students!), then correctness of processes and performances may play as essential a role as the truth value of addition sentences. This might invite more emphasis on checking processes, competing algorithms, and critical implementation in the classroom.

D. Correctness.

At any rate the definition of truth value of (1) seems inseparable from some basic sense of CORRECTNESS. Any psychological model of mathematics learning, at some point should address the issues of truth value and correctness. Not only because mathematics is aimed to truth and based on it. The (psychological) definition of truth and /or correctness will have utilities on the approach to the mathematics teaching. The existing models of cognitive mind, and specifically with computers as medium, seem to have much trouble with these concepts (see Bobrow). The logicians, on the other hand, and rightfully, concentrate on the mathematical definition of truth (Tarski, Quine, Fregê) with little implications on the cognitive components. But the field is not empty. the new model HEIKR, which is describable in mathematical form (Shammas,f) enables us to define (Shammas,c,e) some aspects of CORRECTNESS rather precisely. Consequently an attempt is made to define the truth value of primitive propositions (Shammas,d). Preliminary investigations (Shammas,g) give some hope that this model might be applicable to our case. But the careful reader has, of course, realized that it is based mainly on theoretical analysis. Much research and data are needed to sieve out facts from fiction.

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E. ARITHMETIC AND ALGEBRA

Student Performance Solving Linear Equations

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This paper has four purposes. First, aspects of a model of student performance for solving linear equations will be discussed. Second, the extant research on student performance for solving linear equations will be surveyed. Third, the research results will be related to a potential model of student performance in solving such equations. Finally, in light of the degree of consonance or dissonance of existing research results with the potential model, research questions will be suggested which need to be addressed in order to elucidate the relevance of the potential model toward an understanding of student performance.

Characterizing the Model.

Much research attention has been paid to algebra in general, and equation solving in particular, since 1970. The research studies, however, which are pertinent to a model of student performance for solving linear equations are often based upon quite different perspectives, and employ a wide range of research techniques. The accumulated bits and pieces of information do not obviously fall together into a consistent whole. The creation of that whole requires that some theoretical structure be identified to support and organize the pieces. As a prelude to this and to the discussion of the research, we attempt to provide an abstract characterization of parts of that whole by citing aspects essential to any potential model of student performance for solving linear equations.

Foremost, we are considering the solving of linear equations in one unknown in the context of school mathematics. The task then is a mathematics problem solving task; given such an equation, one is to find its solution set or a numerical correspondent of the variable such that, if the variable is replaced by its numerical value and the computations are independently performend on each side, an identity results. School mathematics carries this task specification a step further to include the method of writing a sequence of equivalent equations. Except for the original equation, each equation in the sequence should follow from its antecedent by means of an acceptable algebraic-logical operation or process, with acceptability determined either implicitly or explicitly in terms of mathematical logical principles or skillful manipulations of proficient solvers such as teachers.

The reference to proficient solvers suggests that modeling of correct performance may be very important to the study of correct performance. A performance model must, therefore, include considerations of learning as students progress from one level of sophistication to another. At the same time a useful model should also provide a backdrop for organizing and discussing errors. As progress is made in this direction, it will become obvious that the school mathematics view of the task is inadequate, being too coarse in some ways to caputre the thinking patterns of the solvers. Indeed, we must seek perspectives more refined in terms of the task and the solver. In particular, it may at times be necessary to consider equations merely as strings of symbols and to study the cognitive processes whereby solvers interpret or maniupulate such symbols. This suggests the appropriateness of enhancing or supplementing the mathematical - school mathematics view of the equation solving task with constructs from related areas such as cognitive psychology, theories of human problem-solving, and performance models in artificial intelligence.

A model of student equation solving performance, then, should take into account a multitude of factors. It must acknowledge different levels of awareness on the part of human solvers, making a distinctions between abilities to use various ideas, to express such ideas verbally, and to provide justification for their use. Closely tied to this is the existence of different types of knowledge: the mathematics knowledge which provides the basic characterization of the task, the perceptual and conceptual knowledge about notation and symbolism which gives mathematical interpretation to the latter, and knowledge of the appropriate use of acceptable operations and processes with the ability to measure progress toward the goal. Thus, the model should be a multi-level account ranging across (a) perception and interpretation of algebraic symbolism, (b) conceptual understanding of the problem-solving task, (c) application of intellectual operations and processes, and (d) the development of strategies and general methods for solving any equation of a particular type.

Reviewing the Research.

Historically, a popular approach for investigating students' equation solving performance was to observe either average test scores or average performance rates for selected items, then to seek possible causes (e.g. Monroe 1915a, 1915b; Davis and Cooney, 1977; Carpenter, et.al., 1978; Carpenter, et. al., 1980). Indications are clear that the larger the number of steps, the less likely students are to correctly solve the equation. Possible difficulties are with combining like terms, transposing terms, doing computation, or in understanding fundamental concepts such as "variable" or "equation".

Although Thorndike, et.al., (1928) separated understanding of an equation as an expression of a certain relationship from the finding of solutions via mathematical manipulations, it is only recently that researchers have become interested in meaning for fundamental concepts (Davis, 1975; Kieran 1979, 1980; Herscovics and Kieran, 1980; Matz, Note 1). Students, in general, seem not to develop a complete understanding of equality but rather take equations as cues to perform a manipulation or to write down an answer. Kieran (1980) suggests that teaching needs to be altered toward improving students' understanding.

Wagner (1981) examined understanding of the idea of equivalent equations by using pairs of such equations, varying the letter for the variable, but otherwise keeping both equations structurally identical. Students who identified the equivalence were said to conserve. Conservation increased with age across 12-, 14-, and 17-year-olds with a significant correlation between conserving and having had algebra.

Thus, the research evidence shows that students do not necessarily develop the meaning of equality intended by mathematicians. To the extent that their interpretations affect performance, it is a challenge to see how a model of students' equation solving performance might be sensitive to these.

Simulation is yet another way to study equation solving performance. Bundy (Note 2) and Bundy and Welham (Note 3) used operations based upon axioms and other algebraic principles, to create an equation solving program with three phases, attraction, collection, and isolation. Heller and Greeno (1979) pointed out that knowledge of Bundy's three phases is not sufficient, but that there must also be some guiding process.

Carry, et.al. (Note 4) based some of their investigation of equation solving skill on Bundy's work. Different strategies were observed to be used for solving particular equations and individual inconsistencies in choice of strategy for solving mathematically similar equations were found. They also identified three error categories: operator, applicability, and execution. Other studies of errors (e.g. Davis and Cooney, 1977; Matz, Note 1; Rugg and Clark, 1918) seem basically consistent with Carry, et.al.

To date, there seems to be enough of an accumulation of research evidence and knowledge about equation solving performance to contrast effective and errored performance. This has the potential for bringing into relief the potential interference among the concepts, principles, and procedures taught relative to solving equations. A model of student equation solving performance should take these into account.

Modeling Student Performance.

Due to space limitations, only a brief sketch of the model is given here. A paper with an expanded, more detailed, account will be distributed at the time of the presentation.

The proposed model is multi-level ranging from perceptual interpretation of the symbolic stimuli used to express an equation, to higher-level, more abstract, concepts and their associated cognitive processes. We introduce the term near features to refer to those aspects which are situation specific and closely tied to the surface symbolism of an equation, while remote features exist only after abstraction has occurred. Remote features encompass the deep structure of an algebraic situation.

Again regarding terminology, we make a distinction between our use of the terms process and operation. For the latter we refer to Berlyne's (1965) model for directed thinking alternating situational and transformational thoughts. The transformations of Berlyne are derived from observations of one kind of stimulus situation being systematically replaced by another kind of stimulus situation. Operation will be used when the action taken by the solver is guided by a single transformational thought or, at least by relatively few transformational thoughts. Intuitively, we are trying to capture the simplest type of alteration that solvers might make as they write one step following another. An operation may still be somewhat complex, however, since a student might error in performing the tactical manipulations of the operation.

Process, on the other hand, will be used to denote a connected sequence of operations. Here we look for episodes of performance which have an integrity of their own. Such sequences are often aimed at attaining strategic subgoals like "removing parentheses" or "getting the x'es to one side." A process may be totally determined by an algorithm; that is, the specific operations as well as the order of those operations are given. However, we allow for cases where decisions are made selecting applicable operations.

To avoid semantic issues, we also allow a process to be based upon a single operation. Furthermore, this convention recognizes an important learning phenomena. Aside from automaticity which speeds up the execution of a sequence of operations, students frequently learn to collapse a familiar sequence of several operations into one of fewer operations; the operations in the new sequence being, in part or totally, distinct from those in the original sequence. Frequent use of a process which might have been developed in association with remote features can lead to the derivation of new transformational thoughts as indicated by Berlyne (1965). The process, seen in a situation specific context, might be re-defined in terms of near features. Typical of this is the development of a transposition process, "moving a term from one side of an equation to the other with a change in sign."

Description of the model follows the solver's flow of attention, first considering identification and interpretation of features ranging from near to remote until a highest level of abstraction/interpretation, depending upon the individual solver, is attained. The flow of attention is then reversed. In passing from higher to lower levels, each subsequent level represents an expansion or redefinition of the processes selected at the next higher level. This continues to a stage which guides the tactical manipulations of observable behavior. A feedback monitoring or evaluation mechanism is also included to account for judgements which alter intentions and produce behavior different from that expected. The above outlines the nature of a basic episode within the overall equation solving performance. This structure, then, is used recursively, as necessary, by the solver until his goal is reached.

Illustrations using the model and discussion of relationships to existing research will be included in the presentation and the full paper.

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ADDITIVE AND SUBTRACTIVE ASPECTS OF THE COMPARISON RELATIONSHIP

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The aim of this paper is to present a number of results obtained in our work with children, concerning the development of arithmetic knowledge by means of research on additive problems. Among them we undertook the study of addition and subtraction problems - when comparison relationship between two measures or states is involved.

Children's achievement and understanding of arithmetic concepts and relationships are a function of several varieBles, to - mention the most important : problem structure, number size, semantic features, amount and form of instruction and cognitive development. Most of available evidence suggests that the semantic structure of a problem is much more important than syntax in determining the processes that children used in their solutions.(Vergnaud & Durand 1976;Vergnaud 1979,1981; Carpenter & Moser 1979;Greeno 1979; Fuson 1981; Nesher & Greeno 1981). These researchers have adopted several classification schemes to characterize additive problems - along dimensions that seem to be the most productive in distinguishing between important differences in how children solve different kinds of problems.

In reference to components and type of problems, structure - analysis and designating terminology our general background is - based on the work done by G.Vergnaud.Our problems follow the criteria proposed by him in 1979 :

- presence or absence of an inclusion relationship,
- existence or not of a temporal relationship, and
- all elements are measures (natural numbers) or one element is not a measure (relative number).

These three main criteria lead to three categories of simple additive problems : a) composition of two measures to produce a third one; b) a transformation links two measures; and c) a static relationship links two measures . Our research concerns only categories b) and c).

The first one (b above) we designated as α or dynamic problems.

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There is an initial state or measure which is followed by an action or transformation (time is involved) which results in a final state. They underlie an inclusion relationship, a temporal transformation and one of the elements is a relative number.

The second one designated by us as β or comparison problems. This category involves the comparison of two distinct or disjoint - sets. Since one referent set is compared to the other, the third - entity is the difference or amount by which the larger or smaller set exceeds or it is less than the second entity. There is no inclusion relationship, no temporal transformation, but one of the elements is a relative number.

Within each one of both categories of problems, there are six types of different situations depending upon which quantity is the unknown and whether the introduced relationship is negative (-) or positive (+) .

EXPERIMENTAL PLAN

Variables

Four variables were considered :

1) presence or absence of a temporal relationship : dynamic (type α) or comparison (type β) problems;

2) unknown is either the initial or referent set (I), final or compared set (F), or the relationship (R);

3) the sign of the relationship : positive(+) or negative (-);

4) the absolute value of the relative number: 2,3 or 4 .

Taking into account these four variables leads to 36 possible different dynamic and comparison problems.

Sample

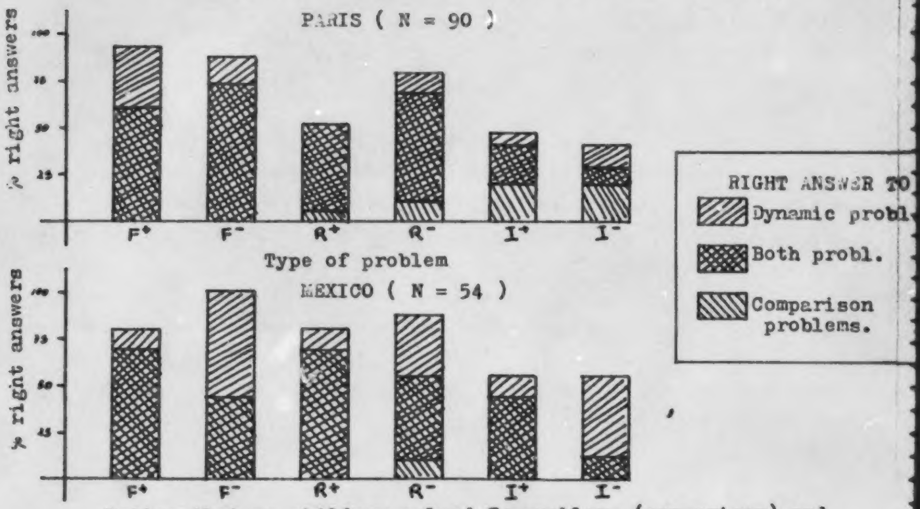
90 french and 54 mexican elementary school children (6 to 9 year old) with whom three groups were formed. Interviews were individual and tape-recorded. During the test the child had at his disposal 20 plastic tokens of the same color and 20 glass marbles.

EXPERIMENTAL RESULTS

1) Comparison between the two categories of problems

Analysis was made with crossed tables of correct and wrong - answers between homologous problems. Percentage of right answers can be seen in Table 1 , which shows clearly that dynamic problems are easer to solve only when the unknown is the final state. For the others positions of unknown the difficulty is similar between comparison and dynamic problems .

TABLE 1. Comparison between the two categories of problems

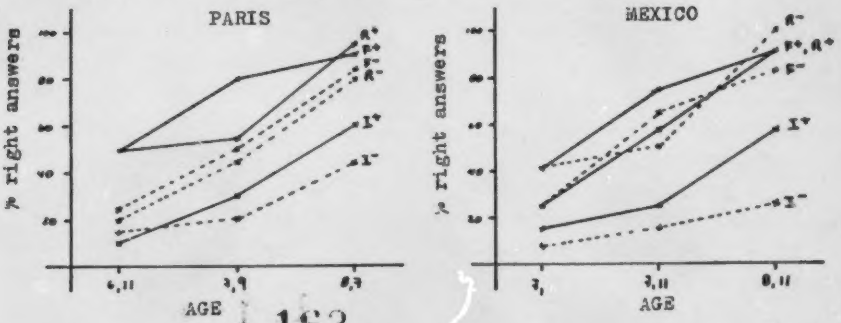


Notice that no children solved F_β problems (comparison) and fail F_α problems (dynamic). Considering the R problems, we observe rather a correlation than a hierarchy. I problems (find the initial or referent set) data indicates that both, dynamic and comparison problems are the most difficult ones.

2) Analysis of comparison problems (β)

In Table 2 we show detailed results for each problem and each group.

TABLE 2 . Percentage of right answers to Comparison Problems



Progression of right answers related to age levels is observed. Besides these overall results we find some difference between the groups studied here, concerning the difficulty between P and R problems.However as stated before by other studies, I problems seem to be of the greatest difficulty.

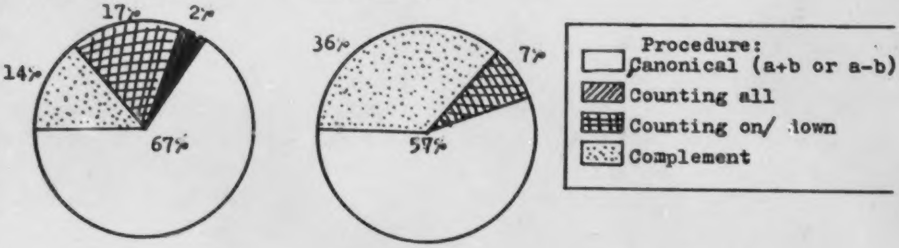
The value of the relative number does not show any significant difference for 2,3 or 4.

3) Solving procedures

Among the solving procedures used by children in our groups we found three main types : counting, complement, and canonic procedures. Canonic procedure is the most often used in all problems for the two samples, as viewed in Table 3

TABLE 3 . Solving Procedures

PARIS (N=161) MEXICO (N=107)



For french sample, counting procedures : counting-all, counting-on,counting-down (Fuson 1981) are used in second term, and among them only the youngest children employed counting-all. Complement procedure is the least produced in this same sample.

For mexican children, complement process is used in second term and in the third place counting procedure . We find interesting that mexican children did not use counting-all among counting procedures.

4) Wrong answers

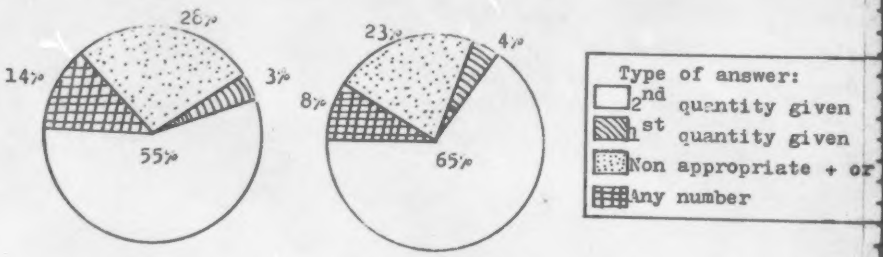
Our results concerning cases of wrong answers are shown in Table 4.We would like to comment only the two types of failures that appeared the most in both of our groups.

Within the first type of wrong answer children give one of the quantities mentioned in the problem,generality the second one uttered, without any numerical calculus. We think that the analysis of errors can provide a report concerning the treatment of the underlying structure of problem and above certain solution strategies. In the case of F and I problems the difficulty for children could be that they consider the relative number as a measure or state and not as a relationship . For R problems subjects give the second number (compared set) because of an incomplet comparison between the entities disables them to find the corresponding relative number.

TABLE 4 . Wrong answers

PARIS (N=179)

MEXICO (N=109)



The second type of mistake produced, only in R^+ and I problems, was to establish a non appropriated relationship of entities. For instance, if the word 'more' was mentioned in the problems subjects performed an addition (when the correct strategy required to inverse the relationship and make a subtraction); and in the case where the word 'less' appeared they made a subtraction (the solving procedure consisting of inverting the relationship and perform an addition).

We find that important differences underlie the types of errors observed, which we think should be undertaken seriously in further research.

FINAL THOUGHTS

Our study provides some experimental results concerning the development of arithmetic knowledge in additive problems, mainly regarding the acquisition process of comparison relationship.

Yet we would like to pointed out that, even if developmental levels in the understanding of additive problems have been proposed in several studies, the present data suggest that these levels are not so clear as one could wish and that further research is required.

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DYNAMICAL MAZES AND ADDITION MODULO 3

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⁂

Introduction

We think that the use of a non-verbal communication device can favour the acquisition of logico-mathematical notions through some built-in constraints which focus the child's attention upon the subject chosen by the teacher, avoiding the problems caused by a lack of verbal communication. Results have been previously presented for children aged 6 to 12 (LOWENTHAL, 1980; LOWENTHAL and MARCQ, 1980; 1981). In this paper, we describe how we used this technique to let 2nd graders (7 to 8 years old) discover the notion of equivalence classes modulo 3, and its properties.

Material

We used as non-verbal communication device the dynamical mazes conceived by COHORS-FRESENBORG (1978) and described by us in this volume (LOWENTHAL and MARCQ, 1982).

Method

We used two approaches with the children. Firstly, we presented the diagram of a network, such as that shown in figure 1, we let the children use the concrete material and build the maze corresponding to this diagram and we then asked them to let trains, numbered in increasing order (train number 0, train number 1, ...), go through the maze.

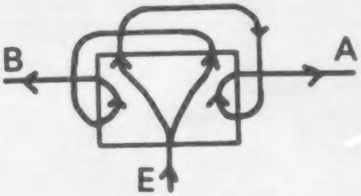

Figure 1.

We also asked them to invent a network "to sort the numbers by n". The network corresponding to the diagram shown in figure 1 can be used to sort the numbers "by 2". In this part of our research, we were more interested in what the children would do after the numbers have been "distributed in classes" among the exits, than by "How can a child predict to which class a number belongs ?" (see : LOWENTHAL and MARCQ, 1982).

All the networks used in this research enabled the user to define "countings" : the numbers (or trains) were equally distributed to the different exits. This defined equivalence classes modulo 2, 4, ... We used only networks defining "coutings" (or equivalence classes) by 2, 4 and 3, in this order. The children knew that it was possible to define networks which did not answer a "counting" process.

Results

a) Build a network for a counting by 4.

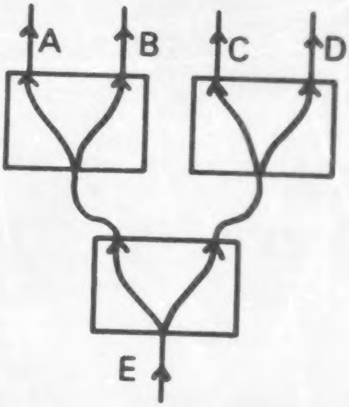

Figure 2.

The children had already used the mazes previously : they had learned to reproduce a diagram with concrete material and to analyze its properties. We now asked them to invent and build a network "for countings by 4" and to distinguish between the multiples of 4 and the sets 4N+1, 4N+2 and 4N+3. 8 seven year olds (out of 20) succeeded to create the network shown in figure 2. This network is not the only solution. There is one which requires 3 switches : one ordinary one and two flip-flops; these elements were immediately available to each child, but the solution is tedious. They chose an easier way to reach a solution which requires 3 flip-flops, but they only had two in their bag of concrete material. They thus had to feel the necessity to use (mentally) a third one and then they had to get hold of it in a reserve bag !

By placing all the switches shown in figure 2 in a starting position : "open to the left" and by letting several "trains" consecutively run through the maze without interfering by changing (by hand) the intermediate positions of the switches, the children observed the following succession of "exits" (outputs) : ACBD ACBD ACBD A...

We told them to associate to each train an order number (i.e. the train number) and to let these numbers run (in increasing order) through the maze. The children then noticed that they obtained the following table :
A : 0, 4, 8, 12, ...
B : 2, 6,10, 14, ...
C : 1, 5, 9, 13, ...
D : 3, 7,11, 15, ...

The children used the ordinal aspect of each number and succeeded thus to distinguish 4 classes. Furthermore, within each class the interval between two consecutive numbers is always the same : 4, the number of classes. Finally, when they let only a finite number of trains run through the maze (a cardinal number n belonging to 4N + 3), the children noticed that there was "an equal number of trains for each exit". They did not explicitly state that this number is (n+1)/4. The regularities of the mechanism enabled the children to establish some connections between the ordinal aspect of numbers, the intervals between them and the cardinal aspect of numbers.

b) Use a diagram and discover properties of classes.

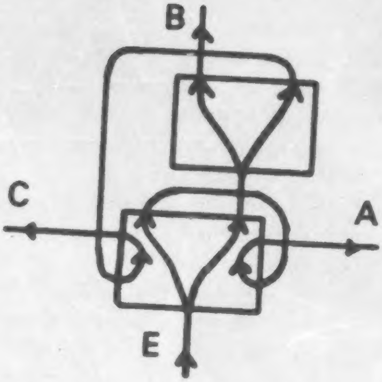

Figure 3.

We presented the diagram shown in figure 3 to our pupils (20 second graders). We had purposely chosen to let them work by groups of 2 in order to favour all possible interactions. Each group received a photocopy of the diagram which was simultaneously projected on a screen using an overhead projector. The screen was used for discussions between representatives of different groups. Some children started to use their photocopy as recording sheet : they wrote the train numbers next to the exit (A, B or C) through which they came out of the maze. The first (train) number they used was 1.

These children had thus created three sets of numbers, 3N+1, 3N+2 and 3N, but they had not given a name to each set. Nevertheless they observed that each set "is a counting by 3" and that if both switches are at the start "open to the left", then "C gives a counting by 3 starting with 3, A gives a counting by 3 starting with 1 and B gives a counting by 3 starting with 2". A child added : "C : it is the multiples of 3, A : it is a counting by 3 but not the multiples of 3, B : it is a counting by 3 but not the multiples of 3". We told them that they had given the same name to A and B, and asked : "Are they the same ?". A child said that they were different since one started by 1 and the other by 2. Another child then added : "A, it is the first after a multiple and B is the next one".

The children used this network in the same way as that shown in figure 2. They discovered that numbers come out of exit A, others out of B and others out of C.

We then asked them to pick a number "in A" and a number "in B". They chose small numbers : 4 and 11. They were asked to add these numbers and could easily check that the sum, 15, is "in C". We repeated this, asking them to pick first a number in B and then one in A. A boy immediately suggested that the sum must be in C "since it is the case for A+B". We asked them to pick two big numbers in A. They came up with 13 and 22. The sum, 35, was larger than any of the numbers they had let go through the maze : no experimental data was available. A girl announced : "It must be in B, you simply drop the tens and you have 5 : that is in B". For the next addition, we chose numbers leading to 48. The same girl announced : "It must be in B". Other children checked, using the maze, and noted : "No, it is in C". They kept track of the situation (exit used) for each number, from 1 to 48. As last computation we chose to let them add two numbers belonging to B : 23 and 14. The result is 37 and it has been established, using the maze, that 37 belongs to A. To our question "Why ?" a child answered "37 is 33 + 4".

Discussion

The material we used enabled some second graders to "sort numbers" : the built-in constraints forced them to become conscious of the order of passage and to establish a relationship between ordinal, interval and cardinal.

This non-verbal communication device enables the children to approach easily the structural problems associated with equivalence classes modulo 3, and with their addition.

In the last result mentioned above (23 + 14 = 37 and 37 belongs to A) the child could have said : "Why ? Because 37 is 36 + 1, 36 is known to be in C, so the next one must be in A". In fact she did something quite different. We think that she decomposed 23 and 14 in, respectively, 21 + 2 and 12 + 2, that she added 21 and 12 to obtain 33 and then wanted to add 2 + 2. Should this be the case, this child would have used the subjacent structure of equivalence classes without mentioning this explicitly. We believe that a few more exercises with these children, using the same diagram, could lead them to verbally state all the rules of addition modulo 3. One should mention here that the 8 children (out of 20) who react positively to these exercices have spontaneously detached themselves from the concrete material : after a certain time they mostly operate upon the notations they have been laid to create (by the built-in constraints of the material). We think that it will soon become possible to approach with these children addition modulo 5, as a generalization of addition modulo 3, without the use of any concrete material, but simply with an abstract machine "which sorts the numbers according to a counting by 5".

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RESEARCH ON THE APPROPRIATION OF THE ADDITIVE GROUP OF DIRECTED NUMBERS.

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1. Introduction.

Considering relational calculus as deduction of new relationships obtained by transforming and composing recognised or accepted relationships, let us give the following definition :

An additive problem is a problem whose solution needs a relational calculus involving only additions and subtractions. Additions and subtractions operate on sets of arithmetical numbers which are either results of measures or absolute values of directed numbers which are associated with transformations, directed states and comparisons. Relational calculus is performed at the level of representations among which figure the algebraic representations that are the equations of the types :

$x + b = c$, $a + x = c$ and $a + b = x$

where a, b and c are directed numbers.

Additive problems formed the subject of different classifications (Moser, Nesher, Vergnaud, Marthe), the last being an extension of Vergnaud's classification to which we refer in this paper.

2. Representations.

Vergnaud's classification consists of several categories. He gave different representations of these categories, some of them being relational diagrams. If we consider that symbolic representations should help students, then further explanations must be given.

If a representation (as a system of signs) linked to a category is applied in another category, it alters

- on the one hand, its nature by concealing the difference of statute of the different numbers involved in the analysis of numerical relationships (for instance, a state - measure - is a positive number, while a transformation is a positive or negative number),
- on the other hand, its way of functioning, the composition of symbols (numbers, arrows, position on the page, ...) depending on the user's reasoning on the diagram.

Thus, when a representation linked to a category is used for another category, it may be in more than one way, that is to say it is not stable.

The need to change a representation as a symbolic system (arrow-diagram, Euler-Venn diagram, relational diagram, ...) when the category is changed results from the consideration of the stability of the representation.

3. Experimental scheme.

The acquisition of knowledge in the conceptual field of additive structures is an object of study so complex that several methods are required to tackle it , each of them permitting the analysis of certain points better than the other. In the proposed experimental scheme, tests, experiments in class-rooms and works in groups have been considered. The children involved in these experiments are 11 - 16 years old (1st - 4th forms). The proposed problems belong to categories for which the additive group of directed numbers may be considered.

3. 1. Standardized test.

The problems of the standardized test belong to the three categories ttt (composition of transformations t between states - measures -), STS (transformation T linking two directed states S), SSS (composition of directed states) and have for algebraic representation the equation $x + b = c$, the absolute value of the first datum b being greater than the absolute value of the second datum c. An estimate for two types of the values of the variables was carried out : either they have the same sign or they have opposite signs. The context was always taken in money problems. A great many teen-agers underwent the test. The results are reproducible with other groups of children. They are the following :

- the research of the unknown is more difficult when the data have opposite signs than when they have the same sign,
- when the data have opposite signs, the research of the initial directed state, knowing the transformation and the final directed state, is easier than that of the first directed state, knowing the second directed state and the compound directed state, which itself is easier than that of the first transformation, knowing the second transformation and the compound transformation,

- the research of the first transformation, knowing the second transformation and the compound transformation, reveals an important and lasting obstacle when the given transformations have opposite signs,
- the more frequent fault found in the two versions (opposite signs, same sign) is the composition of the data, when the additive inverse of the relevant datum is taken and composed with the other datum.

Thus we must consider that even at the end of the fourth form, the concept of the group of directed numbers is not yet operational.

Though this experiment is limited with regard to the choice of categories, it shows that Vergnaud's classification is relevant for the study of the relative complexity of the different classes of additive problems, at least for the set of these three categories.

3. 2. Didactic sequence.

Among other things, the objectives of the didactic sequence are the introduction of directed numbers in two ways (directed state and transformation) and the introduction of the addition of directed numbers in two ways (composition of transformations, composition of directed states and transformations).

In the context of the Loire (a river in France), a journey is a transformation and a place is a directed state. A place is marked by a point along the Loire and a journey is represented by an arrow going from the starting-point to the destination.

The choice of the context of places and journeys along the Loire which allow us to give new representations with a view to introduce relational diagrams was guided, not only by the study of the notion of representation, but also by the study of the notion of the real line which is beyond the single question of additive problems at the level of the research to devote to the algebrisation of the line.

Three tests preceded and led up to these introductions (directed numbers and addition of directed numbers).

The first concerns the finding of the destination of a car starting from a place situated at a given distance from Orléans (a town situated on the Loire) and making a journey along the Loire, the distance of the journey being known. It was estimated that a destination would be more easily found when the car moves off from Orléans than when the car moves nearer to Orléans or goes through Orléans (this being linked with the fact that the composition of journeys of same direction is easier than the composition of journeys of opposite directions). For one of the two problems, the conclusion is contrary to the expected outcome owing to the fact that the solution of the problem is peculiar to the chosen data and the configuration of the map. This first test is followed by the introduction of directed numbers as directed states and transformations.

The second concerns the finding of the first journey knowing the second journey and the direct journey, the directions of these journeys being opposite and the distance covered during the second journey being greater than the distance covered during the direct journey (problem of the category TTT, composition of transformations between directed states; in fact a transformation T is a transformation t). Although a map goes with the problems TTT, these are as difficult as the problem ttt which corresponds to them in the standardized test. This second test is followed by the introduction of the addition of directed numbers as a composition of transformations.

The third concerns the finding of the second journey knowing the starting-point of the first journey, the first journey, the third journey and the destination of the third journey (two problems ETTTE). The ways of solving are very varied and the percentage of successful results is low. One of the problems appears more difficult than the other, the data corresponding neither to distances between towns nor to towns marked on the map. The third test is followed by the introduction of the addition of directed numbers as a composition of directed states and transformations.

After the didactic sequence, a post-test was given. A slight improvement was noted for the problems TTT of the didactic sequence but not for the corresponding problem ttt of the standardized test, the use of drawing journeys seeming relatively effective as representation.

Let us point out that this didactic sequence involves only the first and second forms.

3. 3. Recording of groups.

Groups of four children (1st - 4th forms) were tape-recorded. Two series of recordings were realized :

- the first with problems in different contexts where it concerns the finding of the second transformation, knowing the first transformation and the compound transformation, these transformations having opposite signs, the magnitude of the first transformation being

greater than the magnitude of the compound transformation (1st - 3rd forms),
- the second with the problems TTT and ETTTE of the didactic sequence with students (2nd - 4th forms) who did not take part in it.

For the first series, when the given transformations have opposite signs, the progress of successful results is slow with the first three problems, in spite of the intervention of the relational diagram. The research of an elementary transformation knowing the other elementary transformation and the compound transformation (these having opposite signs) remains a very difficult problem even with the aid of a relational diagram. With regard to the second series, the use of a relational diagram appears to be less effective than the use of the representation of journeys on the map.

In the second series the effect of the works in groups appears to be positive for the problems ETTTE where the children come to the three problems ETE : determination of the destination of the first journey, determination of the starting-point of the third journey and determination of the second journey. Success goes principally through this type of solution, therefore without direct work on transformations. The other ways of solution disappear for the most part. When the two problems are treated differently in the didactical sequence, they are treated in the same manner in the work of groups.

4. Conclusion.

The hierarchy of conceptual difficulty of the concepts of transformation and of directed state determines the behaviour of the children in the solution of additive problems for which the introduction of the additive group of directed numbers may be considered. At least on the investigated part of Vergnaud's classification, it appears relevant for the study of the relative complexity of additive problems.

The main obstacles against which the students come up are the following :
- the first is the research of one elementary transformation knowing the other elementary transformation and the compound transformation, these having opposite signs; this obstacle is linked to a particular category but is not whithout connection with the two following,
- the second is the passage from the set of whole numbers fitted with addition and subtraction to the structure of the group of directed numbers; it appears principally when the children are unsuccessful

in composing the data when the canonical procedure leading to success consists in taking the additive inverse of the appropriate datum and in composing it with the other datum (what forms a problem is the extension of addition and subtraction of whole numbers to the set of directed numbers),

- the third is the identification of a relative state and of a transformation when equating.

Until these obstacles are cleared, the concept of the group of directed numbers cannot be operational.

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INDIVIDUAL DIFFERENCES IN LEARNING VERBAL PROBLEM SOLVING SKILLS IN ARITHMETIC

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The focus of research in mathematics education at the Wisconsin Center is on children's concepts of addition and subtraction as reflected in their ability to solve verbal problems representing addition and subtraction operations. By identifying the processes that children use to solve problems at different stages of instruction and how differences in instruction affect the acquisition of these processes, our research is attempting to gain a clearer picture of how children learn basic concepts as well as to provide some insights into the development of their problem solving abilities.

The major vehicle for investigation has been a longitudinal study of approximately 100 children in two schools in the Madison, Wisconsin USA area. Begun in September 1978 when the children were just beginning American first grade (mean age: 6 years, 6 months), the study's data collection phase ended in January 1981 when the subjects were in the middle of third grade. A major purpose of the study was to trace the development of strategies that children use to solve verbal one-step problems from a point preceding formal instruction through mastery of computational algorithms for addition and subtraction. The longitudinal design has allowed us to obtain a fairly clear picture of the development of each child as well as to observe some basic differences among children that suggest different patterns of learning problem solving skills. Results from the longitudinal study have been presented at earlier meetings of PME (Carpenter & Moser, 1979; Carpenter, 1980; Moser, 1980; Moser, 1981). All of those presentations can be characterized as dealing with analyses of group data. This paper differs in that the emphasis is upon individual performance over a two and a half year period.

The basic dependent measure of the study is a child's performance on a set of verbal addition and subtraction problems where the primary method of determining that performance has been through the use of individually administered problem solving interviews. The interviews were given in September, January, and May of each school year, except for the last one in third grade, resulting in eight sessions. The following six problem types constituted the basic problem set.

1. Change/Join: Jacques had 3 pennies. His father gave him 6 more pennies. How many pennies did Jacques have altogether?
2. Change/Separate: Marie had 12 sweets. She gave 7 of them to her friend Colette. How many candies did Marie have left?
3. Combine/Part unknown: There are 8 children in the class. Five of them are boys and the rest are girls. How many girls are in the class?
4. Combine/Whole unkown: Hendrik has 6 red marbles. He also has 8 blue marbles. How many marbles does Hendrik have altogether?
5. Compare: Klaus won 7 prizes at the fair. His sister Inge won 11 prizes. How many more prizes did Inge win than Klaus?
6. Change/Join, change set unknown: Luis has 6 flowers. How many more flowers does he have to put with them to have 9 flowers altogether?

With suitable word changes to make the problems different, yet retaining the same essential semantic characteristics, problems were administered several times during each interview session to allow for manipulation of variables such as number size and use of other materials such as wooden blocks or paper/pencil. The following diagram demonstrates various interview conditions. Numbers in the problem can be described as having the relationship $x + y = z$, $x < y < z$.

		Number size		
		$5 \leq z \leq 9$	$11 \leq z \leq 16$	$27 \leq z \leq 37$
Presence of manipulatives	with	b+	c+	d+ e+
	without	b-	c-	Not. admin.

d: no regrouping required e: regrouping required in calculation

The "b" and "c" number size problems were given for all eight interview sessions except that "b+" problems were omitted from the last three. The "d" and "e" problems were administered in the last four interview sessions only and it was just for these problems that paper and pencil were given to students to use in solving.

Pupil behavior was categorized according to type of model used (if any), correctness, strategy or process used, and errors (if any). General categories are the following:

1. Direct modeling. A use of the manipulatives provided or fingers in which the objects stand for the problem entities. Actions are performed on the objects, where actions generally correspond to the action or relationship described in the problem.

2. Use of counting sequences. Use of the string of counting words, either forward or backward, where the entry point in the sequence is a number other than "one." Counting may proceed in either direction a given number of counts, or until a desired number (usually one of the numbers given in the problem) has been reached. This requires a second counting of some sort as a tracking mechanism, often aided by the use of fingers.
3. Mental operations. This involves a use of memorized number facts, either by a direct recall or by the derivation of a non-memorized fact through manipulation of some other recalled fact. As an example, the fact for 6 + 8 can be derived by determining it to be two more than the easily remembered "doubles" fact of 6 + 6.
4. Use of computational algorithms. Recorded for two-digit "d" and "e" problems only. This category includes the standard algorithms taught in school as well as any "invented" (Moser & Carpenter, 1982) ones that involve considerations of place value. Algorithmic behavior may be exhibited by paper and pencil, or done mentally as was frequently seen in problems in which no regrouping ("d" problems) was required. (Moser, 1981)

Other inappropriate behaviors were also noted, such as guessing, using one of the given numbers in the problem, adding instead of subtracting, or giving no answer at all.

Within the first two categories, there are particular behaviors that can be described as corresponding very closely to the semantic structure of the verbal problem. As an example, if one considers problem type #6, the Change/Join, change set unknown, it can be seen to be solvable by the mathematical oeration of subtraction. Yet, it is additive in character. The direct modeling behavior corresponding to this problem is one we have called Adding On, where the child constructs an initial set to model the smaller given entity of the problem and then adjoins objects to that set, one at a time, until a larger set of numerosity equal to the described size of the larger given entity in the problem has been formed. Counting the number of joined objects gives the solution. The counting-sequence analogue is Counting Up from Given in which a forward counting sequence is entered at a number corresponding to the smaller problem entity and ended at the number corresponding to the larger problem entity. Counting the number of spoken number words in the sequence gives the solution. In contrast, the Comparison problem type is best modeled by the Matching strategy in which two model sets are put in a one-to-one correspondence and the solution determined by counting the excess of the larger over the smaller.

Empirical Findings

In a brief written report such as this, a major difficulty arises in attempting to give the reader a good sense of the totality of results. For example, at the time of the fifth interview in January 1980, each of the six different interview conditions shown in the diagram presented earlier was imposed. For each condition, six separate tasks were given. When this number is multiplied times eight different interview periods, and then multiplied again by almost 100 different subjects, one can begin to appreciate the difficulty of trying to distill all these data into some sort of smaller unit that is understood by a somewhat naive reader. Given these numbers together with the fact that each problem can be solved by a variety of different strategies, the laws of probability alone are almost sufficient to guarantee the conclusion that each individual student profile over the period of the longitudinal study is different from every other one. Nevertheless, even though final data analysis is not completed, some clear patterns of behavior have emerged and will be discussed in the following paragraphs.

An individual student profile will be characterized as a point-by-point look at a child's strategies over the different time periods. It is not possible to give a single descriptor for a child at a single point in time because performance does vary according to problem types. For addition, there was no marked differentiation in performance depending on whether the problem was an active one (Task #1) or a static one (Task #4) and so a single descriptor is possible for this operation. However, for subtraction the situation is quite different and distinct descriptors will be given for the subtractive (Task #2), additive (Task #6), and comparative (Task #5) problem types. For the basic fact number size problems ("b" and "c"), it is possible to compress data to give a single descriptor of behavior. However, a child's behavior on a basic fact problem generally differed from his/her behavior on a two-digit problem of the same type given at the same point in time.

The predominant pattern of behavior is one of a natural progression through increasingly sophisticated levels of abstraction. That is to say, children seem to start with direct modeling, then move to use of counting sequences, which is then followed by the formal use of number facts and computational algorithms. Of course, within this progression, there is a great variability from child to child, and even within one child variability from problem type to problem type. For example, there were a number of children who progressed to use of counting sequences for addition at an earlier point in time than for subtraction. Within subtraction, the additive problem type (Task #6) tended to evoke counting sequences and use of number facts earlier than for the other types. Great differences were also observed in the timing when children moved from one level to the next. At the time of the first interview, some children were already using counting sequences for certain problems and recall of basic facts for others (often smaller number addition problems).

As mentioned earlier, direct modeling and use of counting sequences can be further characterized by their connection to the semantic structure of the verbal problem. As a further example, the Separating From modeling strategy of building an initial large model set and then removing objects one at a time is directly related to the subtractive nature of the Change/Separate (Task # 2) problem type. It is not quite so apparent to detect the influence of problem structure on higher level strategies such as use of mental operations or algorithms. With this observation as background, it is possible to describe two other sub-patterns of behavior that exist within the first pattern described above. The first is the finding that for many children, the choice of use of strategy is structure-governed. In this instance, the child would appear to base his/her decision upon the semantic structure of the problem. In earlier reports of results from the longitudinal study (e.g. Carpenter & Moser, 1979), structure-governed behavior seemed to be a rather general occurence. However, now that results from the entire two and one-half year period are available, it is clear that this determining factor does not influence behavior over an extended period of time for all children. Yet, for those who do give evidence of being influenced by this factor, at least up to the time that they shift to more advanced strategies, those subjects can be characterized as being structure governed. It should be noted that this trait, while not being too readily observable with the smaller number problems in the later interviews, reasserts itself as children become confronted with larger two-digit number problems for which they do not yet have a usable computational algorithm.

In contrast, there are children who can best be described as strategy-governed. This category is most evident in the subtraction behaviors where the problems are clearly different in semantic structure. Children in this group tend to exhibit behavior where it can be inferred that they have come to realize that a particular strategy is appropriate for use, regardless of the semantic structure. For example, the child will use a Counting Up from Given strategy, regardless of problem type. For others, the subtractive Separating From modeling strategy is the choice for solution of all subtraction problems. While the structure-governed behavior tends to appear almost at the very beginning of our study, the strategy-governed behavior is one that evolves over time, giving some support to the conclusion that this latter behavior may be more influenced by direct instruction. It must be noted that there are large differences among individual subjects in terms of the degree to which they are strictly structure- or strategy-governed. Stability of performance was notably lacking among almost all subjects and many would give occassional evidence of being governed by one factor or the other or both.

Concluding Remarks

Space considerations preclude the inclusion of individual profiles which would make some earlier remarks much clearer. In previous reports on this study made to PME, attribution of causality for strategy choice was given to problem structure and to the inventiveness of young children. In a recent paper (Moser & Carpenter, 1982) in which behavior on the two-digit problems was examined more closely, student inventiveness was again seen to play a very important role in the types of strategies employed. Yet, one cannot discount the extremely important influence of direct instruction. In later reports to be prepared by the author and his colleagues at the Wisconsin Center, we hope to pursue this matter in much greater depth.

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CHILDREN'S COUNTING

Jan van den Brink

Introduction

Our current research covers three contents areas: counting, numbers in a context, pocket calculator. The subjects are 4 to 7/8 years olds..,

The first - counting - is submitted here. At the same time an extension of our research method (mutual observation) is presented.

MUTUAL OBSERVATION

It is a big problem to check the reliability of the child's introspection. (In arithmetic problems there is a tendency that questions like 'how did you do it' are answered by displaying the method according to which it should be done).

Mutual observation means reading the protocol notion immediately to the subject, who in this way has an essential part in the research: he understands what the interviewer is looking for and can help him. (Van den Brink, 1981)

Instruction is a field full of suggestions to discover new research methods and techniques and to improve old ones. Mutual observation is such an example (Cp. Balacheff, 1981).

Instruction has been the source of inspiration of the following ideas, beside mere reading of the protocol:

a) simulating knowledge of the child's thought in order to stimulate a response;
b) having the subject guess what the interviewer might have thought he would answer;
c) pretending ignorance;
d) change of roles;
e) conflicts and jokes.

In brief, quite a lot of social techniques can be used to stimulate mutual observation.

COUNTING

Counting is not restricted to quantities. It is a tool in moving, playing, singing, passing away the time, and so on. This kind of counting is most often mere acoustic counting, of which counting quantities can be an application.

Acoustic counting is reciting sequence of numbers in a certain order — a broad definition, since in fact the order need not be the natural one. As little needs it to be started with 'one' nor need units and quantities to be indicated by finger and eye movements.

Phenomena as appear in our research with 4 to 7/8 years olds can be divided into three chapters:

1) the dynamics of acoustic counting (counting and moving)
2) the sound systematics of acoustic counting
3) ways to count quantities

From these chapters we will touch a few subjects:

1) *The dynamics of acoustic counting*

a) monotonous and rhythmic counting.

If children are asked to imitate a sequence of beats, it makes a difference whether it is *monotonous* or *rhythmic* and whether beating is accompanied by counting or not.

In the case of *monotonous* counting there is a tendency to overshoot the mark. It is as though they need a braking path.

In the case of *rhythmic counting* on the contrary, then is a tendency to start counting anew after each period.

Monotony propels counting; it is an important device in learning the sequence of numerals. Stops are not readily accepted.

Rhythms in counting are easily blotted out.

On the other hand they can play a part in forming new units. (Cp. Von Glasersfeld, 1981)

b) Asynchronisms

Another phenomenon is that counting and moving (e.g. tapping) do not go together.

An asynchronism can have various causes:

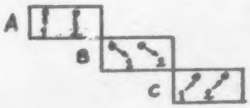
. either the counting sequence is insufficiently mastered: in spite of halterings the movements goes on.

. or the counting sequence is too well mastered: the movements stay behind the counting.

. a third possibility: the lack of any connection between counting and moving.

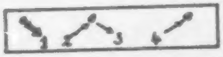
Little children use three control systems to synchronise counting and moving (tapping):

A. tap and numeral simultaneously,
B. first the tap, then the numeral,
C. first the numeral, then the tap

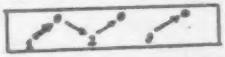


While counting children might change the system which causes an asynchronism. We noticed two kinds

from systen B to C



from system C to B



Another application of counting: the researcher performs a pattern of jumps with the finger between a number of spots to be reproduced by the subject. This task is made easier by counting. Anyway the *number* of jumps is correctly reproduced, though asynchronisms can disturb the reproduction pattern. Rhythmic counting is here more helpful than monotonous counting.

2) *The sound systematics of acoustic counting*

a) The natural number sequence and other acoustic sequences.

For the five years old counting is exclusively reciting the natural number sequence 1, 2, 3,..... For Dutch first graders the natural number sequence is not connected with acoustic counting of tens (10, 20, 30,...). They are two seperate sequences, which acoustically are not closely enough connected. The connection has to be demopstrated by means of abacus.

The acoustic sequence of hundreds 100, 200, 300,... is different. These sounds agree more with those of 1, 2, 3,.... than do 10, 20, 30,..

Familiar sequences such as 10, 20, 30,...; 5, 15, 25, 35,....; 8, 18, 28, 38,... are easily learned by firstgraders by means of sound concordance, with no 'meaning' included. By this way they can help to prepare the meaningful positional system:
Grouping by tens —— the numerals to count a grouped collection are preexistent albeit in another acoustic context (Cp. Kühnel, 1966).

b) Sound units

New numerals to continue counting are formed by

repeating the old sequence

and

choosing a new 'sound unit'

(In Dutch as in German it is 'one and twenty', 'twoo and twenty',... and 'twenty' is to be the new 'sound unit').

Poor counting can mean: omitting the new unit and merely repeating the old sequence. In general after 20 children use 'twenty' in the meaning of a (sound) unit: 'one of twenty' (21), 'two of twenty' (22), 'three of twenty (23),....

As a consequence they continue after 100 by 'one hundred', 'two hundred', ... which are known and substitutes for (not existing) 'one and hundred', 'two and hundred',...

So only one first grader said 105 in the correct way of 'hundred five' while all others said five and hundred or five hundred.

In dictations the different pronounciations can be confronted with each other: what is the difference between 103 and 300?

Choice between two 'sound units' can lead to conflicts in counting. Shift from the counting sound to another sound unit can produce (even with second graders):

24, 25, 35, 36, 136, 137

The prefix hundred causes no troubles but it does so if there is a choice of sound units: 136, 137, 237, ?

3) *ways to count quantities*

With our subjects we noted five ways to count quantities. They are restricted by the various activities that play a part in counting quantities.

Apparently a child has to combine three counting elements

a) the objects to be counted,
b) the numerals to be pronounced,
c) the acts of indicating or moving objects.

It depends on the child which element is stressed.

In our research we distinguish

A *Moving*

For 4/5 years olds counting objects is a task of moving: eye, finger or object. This is in particular stressed if moving collectious (children on the playground) are counted

B *Propelling counting*

Rather than the objects the indicating movements are counted. The counting activity propels the movement.

C *Accompanying counting*

In contradistinction with B the movement can be primary while counting is an accompanying phenomenon.

D *Labeling*

Counting objects is labeling each of them, to wit by means of numerals

E *Proper object counting*

The objects themselves (rather than the movements or the numerals) are the stressed things.

The variety of counting methods migth be the consequence of assiduous attemps to imitate what is understood as counting activities of adults. (Cp. Steffie, 1981; Richards, 1981).

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F. RATIO AND PROPORTION

TURNING THE PLATES —
SIZE PERCEPTION OF RATIONAL NUMBERS
AMONG 9- AND 10-YEAR OLD CHILDREN[1]

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The question of how children perceive the "bigness" of rational numbers is of importance. Addition of the fractions like $\frac{1}{3}$ and $\frac{3}{5}$ does not make sense to a child without a quantitative notion of the addends. Only if children perceive a fraction, given and first appearing to them as an ordered pair of whole numbers, as an entity, is it likely that they will extend their notion of number to include fractions. One measure of children's quantitative notion of rational number is their ability to perceive the relative size of pairs (or sets) of rational numbers; that is, to determine which of two given fractions is less or whether they are equal. This is discussed in (Behr, Post, Wachsmuth, 1982), based upon information acquired in the context of a 16-18 week teaching experiment.

A concrete response mode was used in the present study, conducted during the 1980/81 period of the Rational Number Project[2] still in progress (Behr, Post, Silver, and Mierkiewicz, 1980), to gain information about children's perception of the absolute size of fractions. Subjects were six children in a 4th grade experimental group in DeKalb, five of which were matched with subjects in a 4th grade comparison group of the same school. In addition, data is available from a group of six 5th graders which was the pilot experimental group in 1979/80, and a 5th grade comparison group.

To show the size of a fraction with a concrete embodiment, children were provided with a set of two intersecting plates of red and of green color. These could be turned relatively to each other in a way to show a smaller or bigger amount of the whole circular device shaded green with the restbeing of red color, thus embodying a fractional part of a circular unit. This device actually was recommended as a teaching tool by Leutzinger and Nelson (1980), and was used here to gain information about children's perception of the absolute size of a fraction; the angle measure serving to record the precision of the students' responses. None of the subjects had seen or used this device before; however, its perceptual proximity to the usual circular models used in instruction provided immediate understanding of its functioning.

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[2] The term "rational number", in this paper, is used when emphasizing the number concept of fraction (as opposed to ordered pair).

Students were asked to use these intersecting plates to show each one of the fractions $\frac{1}{3}$, $\frac{1}{5}$, $\frac{3}{5}$, and $\frac{6}{10}$ as fractional part of the circular whole. Questions were posed in the form "Make it so that one-third is green" [as writing $\frac{1}{3}$]. Recorded was the student's behavior, the degree measure of the angle shown, and the student's oral explanation of what s/he thought to do it. Interview data is available from two assessments; the first, half way through and the second, at the end of the teaching experiment.

Children's responses on these items are categorizable into seven categories plus one additional -- don't know. Descriptions and representative responses from children to exemplify the categories follow.

Category UFI (Unit fraction iteration) Subject finds three-fifths for example, by first finding one-fifth and then doing a definite iterative behavior.

To show three-fifths "I moved to one-fifth, then I thought of another one. That would be two-fifths. Then make it one more fifth."

Category PP (Physical Partitioning) Subjects' response is based on the use of imaginary lines which trace unit fractions of the desired fractions. After partition of the whole plate is made, S moves directly to the original fraction.

To show one-third S explains "one-third is like a Y. I drew imaginary lines in my head and spaced them out. All the parts were like [the one] shaded green."

Category E (Recognized equivalence) Subject recognized that the fraction on the display was equivalent to the desired fraction and did not change the display in showing the desired fraction.

Category RU (Reference to a unit fraction, other than 1/2) Subject compares the desired fraction to a known unit fraction.

To show 1/3 the subject says "Can you do that with a quartered plate [showed one-fourth of the plate green]? Probably like that [increasing the angle from the one-quarter position]?"

Category RH (Reference to 1/2) Subject makes a relative comparison of the desired fraction to the known fraction 1/2.

To show 6/10 S explains "See if you make an imaginary line [making 2 halves], that's 5 [tenths] and then that's 6, 7, 8, 9, 10 . Well I had it one-half first and that was wrong which I know. If you make it one-half and then over 1 more, that is 6, and 7, 8, 9, 10 pieces".

Category RIM (Reference to incorrect inventive model). Reference is made to some incorrect concrete model that denotes some divisions (e.g. uses a clock; 5 o'clock = 1/5).

Subject apparently uses the 3 o'clock position to show 3/5 saying "You went like ... here's 10 o'clock, 20 o'clock and I went over to 30 o'clock [3 o'clock] for three-fifths."

Category AD (Addition by denominator difference). Subject uses an addition rule to get from a previous position to the desired position; adds differences (e.g. if previous display was at 1/3, 1/3 + 2 = 1/5, therefore move 2 times from what was 1/3).

Showing one-fifth having one-third displayed the subject explains "Well it was on 3 so I just moved it 2 more times."

Category DK (Don't Know) Subject is unable to solve or gives a random response and says he/she guessed.

A distribution of subjects responses by groups is given in Table 1.

Table 1. Response Frequencies within Categories by Subject Groups

Response Category	5th Grade		4th Grade	
	Exp.	Comparison	Exp.	Comparison
E	1(2.3)[a]	2(5.0)	0	1(2.5)
UFI	10(23.2)	4(10.0)	11(25.6)	2(5.0)
PP	30(69.8)	10(25.0)	28(65.1)	18(45.0)
RU	0	3(7.5)	0	1(2.5)
RH	2(4.7)	4(10.0)	4(9.3)	0
RIM	0	5(12.5)	0	2(5.0)
AD	0	6(15.0)	0	2(5.0)
DK	0	6(15.0)	0	14(35.0)
Totals	43	40	43	40

[a] percent of responses in this category as a percent of the total number of responses in the given group.

The sequence of fractions given was furthermore chosen to investigate whether the student would make a connection to the fraction preciously shown (i.e. $\frac{1}{5}$ must be represented by a smaller angle than $\frac{1}{3}$, $\frac{3}{5}$ can be perceived as $\frac{1}{5}$ and $\frac{1}{5}$ and $\frac{1}{5}$, $\frac{3}{5} = \frac{6}{10}$ so the same display can be used), which would indicate whether s/he already interlinks rational numbers. Observations noted a child's initial direction to show $\frac{1}{5}$ when $\frac{1}{3}$ was showing, for example, and it was asked, how did you know whether to make the green part bigger or smaller? Responses and observable behavior were categorized in three categories:

C^+: subject makes a connection with correct intention;
C^-: subject makes a connection with an incorrect intention;
N: no connection observable.

Subjects' responses and categorizings are shown in Table 2 and Table 3.

Observations and Conclusions

From Table 2 and 3 we gain some insights into the effectiveness of the experimental treatment with respect to the criterion variable. According to within group mean deviations, the groups were ordered by performance such that the 5th and 4th grade experimental groups were higher than either of the matched comparison groups, with one exception: in the 6/10 task of the final interview, the 5th grade comparison group outperformed the 4th grade experimental group. In most cases, the difference in favor of the experimental groups was substantial. These results

Table 2. Subject Response Data: 5th grade 1979/80
experimental and comparison groups

Subject	Mid-Experiment Assessment				Post-Experiment Assessment			
	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)
Experimental Group								
1(H)[a]	70[b] (-50)[c] PP -	68 (-4) PP C+	88 (-128) UFI C+	202 (-14) UFI N	100 (-20) PP -	50 (-22) PP C+	158 (-58) UFI N	225 (+9) PP N
2(H)	106 (-14) PP -	64 (-8) PP C+	246 (30) PP N	207 (-9) RH N				
3(M)	97 (-23) PP -	82 (+10) PP C+	115 (-101) PP N	109 (-1[illegible]) PP N	105 (-15) PP -	72 (0) PP C+	210 (-6) PP C+	200 (-16) E N
4(M)	118 (-2) PP -	62 (-10) PP C+	182 (-34) UFI C+	163 (-53) UFI N	100 (-20) PP -	60 (-12) PP C+	220 (+4) UFI C+	205 (-11) RH N
5(L)	107 (-13) PP -	PP C+	142 (-74) PP C+	202 (-14) PP C-	115 (-5) PP -	60 (-12) PP C+	215 (-1) UFI C+	150 (-66) PP N
6(L)	120 (0) PP -	74 (+2) PP C+	149 (-67) UFI C+	131 (-85) UFI N	128 (+8) PP -	70 (-2) PP C+	228 (+12) - -	240 (+24) UFI N
$\bar{X}$	(17.0)[d]	(6.8) [100][e]	(72.3) [67]	(47.0) [0]	(13.6)	(9.6) [100]	(16.2) [60]	(25.2) [0]
Comparison Group								
1(H)	110 (-10) RIM -	260 (+188) AD C-	279 (+63) RH C+	358 (+142) DK C-	210 (+90) AD -	220 (+148) AD C-	300 (+84) RH N	300 (+84) E C+
2(H)	80 (-40) RU -	100 (+28) RU N	290 (+74) RU N	305 (+89) DK N	110 (-10) PP -	80 (+8) PP C+	255 (+39) PP N	255 (39) E C+
3(M)	104 (-16) PP -	62 (-10) PP N	250 (+34) UFI C+	220 (+4) UFI N	139 (+19) PP -	65 (-7) PP C+	246 (+30) UFI N	228 (+12) PP N
4(M)	85 (-35) DK -	159 (+87) AD C-	64 (-152) DK C-	189 (-27) DK N	20 (-100) RIM -	35 (-37) AD C-	150 (-66) PP N	152 (-64) PP N
5(L)	37 (-83) RH -	162 (+90) RIM C-	75 (-141) RIM N	179 (-37) RIM N	22 (-98) RIM -	34 (-38) AD C-	99 (-117) UFI C+	169 (-47) DK N
$\bar{X}$	(36.8)	(80.6) [0]	(92.8) [40]	(59.8) [0]	(63.4)	(47.6) [40]	(67.2) [20]	(49.2) [40]

[a] Denotes mathematics achievement level: High, middle, low.

[b] Degree measure of subjects trial.

[c] Deviation of subjects trial from true measure.

[d] Mean deviation.

[e] Percent of subjects who made positive connection.

Table 3. Subject Response Data: 4th grade 1980/81
experimental and comparison groups

Subject	Mid-Experiment Assessment				Post-Experiment Assessment			
	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)
Experimental Group								
1(H)	89 (-31) PP -	90 (+18) PP C+	115 (-101) PP N	120 (-96) PP N	103 (-17) PP -	77 (+5) PP N	142 (-74) UFI C+	118 (-98) UFI N
2(H)	130 (+10) PP -	68 (-4) RH N	142 (-74) UFI N	173 (-43) UFI N	110 (-10) PP -	70 (-2) PP C+	154 (-62) RH C+	N
3(M)	115 (-5) PP -	55 (-17) PP C+	145 (-71) UFI C[illegible]	[illegible] ([illegible]29) UFI N	120 (0) PP -	70 (-2) PP N	120 (-96) UFI C+	125 (-91) PP N
4(M)	110 (-10) PP -	80 (+8) PP N	120 (-96) UFI N	[illegible] ([illegible]) PP N	150 (+30) PP -	92 (+20) PP C+	215 (-1) UFI C+	186 (-30) UFI N
5(L)	104 (-16) PP -	55 (-17) PP C+	122 (-94) PP N	178 (-38) RH N	119 (-1) PP -	62 (-10) PP C+	200 (-16) UFI C+	148 (-68) RH N
6(L)	56 (-64) PP -	67 (-5) PP N	145 (-71) PP N	121 (-95) PP N				
$\overline{X}$	(22.7)	(11.5) [50]	(84.5) [17]	(55.3) [0]	(11.6)	(7.8) [60]	(49.8) [100]	(71.8) [0]
Comparison Group								
1(H)	60 (-60) RIM -	80 (+8) AD C-	85 (-131) DK N	- DK N	74 (-46) PP -	83 (+11) PP C+	138 (-78) UFI N	169 (-47) PP N
2(H)	50 (-70) PP -	45 (-27) PP C+	70 (-146) DK C-	80 (-136) RIM N	96 (-24) PP -	54 (-18) PP C+	148 (-68) PP N	153 (-63) UFI N
3(M)	119 (-1) RU -	142 (+70) AD C-	2[illegible] (+14) DK N	64 (-152) DK N	80 (-40) PP -	68 (-4) PP C+	70 (-146) PP N	70 (-146) E C+
4(M)	63 (-57) DK -	37 (-35) PP N	155 (-61) PP N	122 (-94) PP N	93 (-27) PP -	67 (-5) PP C+	193 (-23) PP N	90 (-126) PP N
5(L)	68 (-52) DK -	180 (+108) DK C-	65 (-151) DK C-	142 (-74) DK N	55 (-65) DK -	180 (+108) DK C-	315 (+99) DK C+	305 (+89) DK C-
$\overline{X}$	(48.0)	(49.6) [20]	(100.6) [0]	(114.0) [0]	(40.4)	(29.2) [80]	(82.8) [20]	(94.2) [20]

suggest a positive effect on children's quantitative perception of rational number due to the experimental treatment.

From Table 1 we observe that both the 4th and 5th grade experimental groups used fewer and higher level strategies. Over 65% of the subjects in both experimental groups used the physical partitioning (PP) strategy compared to 25% and 45% in the 5th and 4th grade comparison groups. Worthy to note is the comparatively high percentage of experimental subjects who used the strategy of unit fraction iteration (UFI). The UFI strategy presupposes an understanding of the concept of unit fraction, which can be provided by the more primary PP strategy. Physical partitioning requires the perception of some physical object as a whole. The UFI strategy , on the other hand, does not require making reference to an object or a quantity as a whole. It, in reverse, takes the unit fraction 1/5 as an *entity* (as opposed to an ordered *pair*), and from this is built non-unit fractions, the whole, and fractions greater than 1. Instead of abstracting 3/5, for example, directly from physical part-whole embodiments, UFI, as with 3/5 = 1/5 and 1/5 and 1/5 (i.e. 3 one-fifths) already interlinks rational numbers at a higher order of abstraction. That is, UFI achieves the building of conceptual structures by making connections within the domain of fractions, and thus leads more directly to an abstract thinking of rational number.

Moreover, UFI provides a basic understanding of addition of fractions (with same denominator) which is close to counting-on strategies with whole numbers -- to process 3+4=N, count 3; 4, 5, 6, 7; to process $\frac{3}{5} + \frac{4}{5} = N$, "count" $\frac{3}{5}; \frac{4}{5}, \frac{5}{5}, \frac{6}{6}, \frac{7}{5}$ (exceeding 1 is not a problem with this strategy).

In particular the combination of PP/UFI strategies seems to facilitate knowledge building toward a quantitative understanding of fractions which becomes independent of part-whole (ordered pair) notions, by embedding single concepts in a structure of other concepts, and consolidating prior concepts. This is supported by the superior performance of the two experimental groups.

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Strategies and Errors in Secondary Mathematics - Teaching Modules to Correct Errors in Ratio and Proportion.

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The research of the project "Strategies and Errors in Secondary Mathematics" is based on the results obtained by the earlier CSMS investigations (Hart, 1981a). In that project, a third of the sample (n=3000) who had attempted the test paper containing ratio and proportion problems, produced answers (to the difficult questions) which suggested the use of addition. The child using the "incorrect addition strategy" in enlargement questions would, for example, interpret 5:3 as an increase of 2, so 4 would become 6. The first phase of the current research was to interview a number of children whose written answers suggested that this method had been used, in order to verify that it had been employed. These children were usually in classes of 'average' attaining pupils and not in the group with the lowest level of achievement. They overwhelmingly, however, used repeated addition as a replacement for multiplication when dealing with the easier ratio and proportion questions (Hart, 1981b).

A major aim of SESM (Strategies and Errors in Secondary Mathematics) is the formulation of teaching modules which can be used by the classroom teacher, in order to remedy the identified error. The basic assumption in this formulation is that by identifying the erroneous ideas that appear in the reasoning of the 'error' children, one can provide instruction which will remedy this situation. The first teaching module designed to remedy the 'incorrect addition strategy' in enlargement of figures involved:-

1) Showing the pupils the error inherent in their method
2) The provision of examples which naturally suggested the operation of multiplication rather than addition
3) The provision of examples which highlighted a need to know the result of multiplying non-integers (e.g. 2½ x 1½)
4) A series of lessons on multiplication of fractions
5) The teaching of a method for finding a scale factor given the length of a side and the length of its enlargement.

Three groups of children (n=12) were taught a series of lessons which embodied these ideas (a total of about 200 minutes). The members of each group were chosen because they seemed to use the incorrect addition strategy when faced with difficult enlargement problems. The experiment was designed in the classical pre-test; treatment; immediate post-test; delayed post-test format. On both post-tests the addition strategy was almost entirely absent, however, although considerable success had been achieved on the immediate post-test, by the delayed post-test the questions were not being solved correctly. Of the 12 children in the study, only 7 had been present at all the lessons and of these only one child could be said to be completely successful on the delayed post test. Only one child continued to use the incorrect addition strategy; the errors committed by the others were concerned with the manipulation of fractions. This first phase served to highlight some attributes of "adders". Their grasp of the whole idea of fractions seemed to be tenuous and although they appeared in the short-term to be able to multiply fractions, when they had recently been taught the process, more elementary fractional concepts seemed to be lacking. There also seemed to be a high absentee rate amongst these children, sufficient perhaps to prevent them acquiring a routine methodology for solving ratio and proportion problems when taught in the normal class.

The first teaching module had not attempted to present either of the Proportion algorithms usually taught in British classrooms i.e.

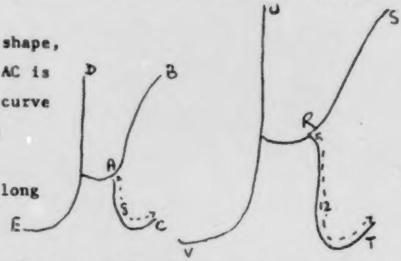
(i) $\frac{x}{a} = \frac{y}{b}$

(ii) the unitary method.

Instead the basic ideas had been that enlargement is brought about by the multiplication of measurements by a scale factor which can be found by solving an equation. The method for multiplying non-integers then being taught. This had proved to be insufficient because the deficiency in the understanding of fractions was rather more widespread than simply the operation of multiplication.

The "addition" children solved easy problems by using repeated addition rather than multiplication and failed to solve harder problems because they employed an incorrect strategy. This incorrect addition strategy could be eradicated through teaching but its replacement by a workable general method was not so easily achieved. Further evidence on the incidence and type of successful methods was obtained by interviewing a small sample (n=9) of children, (aged 12-15) selected by their teachers as being "good at mathematics". The difference between these and the adders was not in their ability to produce a teacher-taught algorithm which could be applied to any question but in their understanding and flexibility in the use of fractions. Let us consider some examples of the flexibility displayed when a question from the CSMS Ratio paper was set:-

These two letters are the same shape, one is larger than the other. AC is 8 units. RT is 12 units. The curve AB is 9 units. How long is the curve RS?
The curve UV is 18 units. How long is the curve DE?



A typical 'adder' would reason 'Eight becomes 12 by adding on 4. So add 4 onto 9 for RS. Answer 13'. The successful girls recently interviewed answered as follows:-

Madeline: 8 to 12. 8 is two thirds of 12, so AB will be 2/3 of RS. So you find half of this which is 4½ and add it onto 9 which makes 13½

Rose: The difference between that is . ... that RT = AC + ½AC so if AB is 9, then RS is 9 plus half AB, which is 4½. Added together to make 13½.
UV is 18, so you have to find 2/3 of it. Working from that way. That is 2/3 of that. You have to find 2/3 of 18 which is 12. Six goes into 18 three times.

Clare: Twelve is two thirds of 18 and 8 is 2/3 of 12 and they're similar. So this K is 2/3 the size of this. Two-thirds of this K and so if 9 is two-thirds, we want to find three thirds. That means that if 9 is two thirds, a third is 4½ so if you add 4½ on to 9 you get 13½.

It is apparent that not even these 'bright', successful pupils handle multiplication of fractions with ease but prefer to translate the question into addition. The second teaching module was designed in a similar way to the first but the phase devoted to the teaching of multiplication of fractions was replaced by one which introduced the use of the calculator. A first trial of this module with a group of 13 year old "adders" showed considerable success on the immediate post-test and a display of confidence not apparent in the first module trials. The delayed post-test was given some 13 weeks later and by then the children had forgotten most of the instruction. After brief prompting what they remembered differed from the taught methods and it was this child-remembered form of the use of the calculator that was used in the next teaching module.

It is intended that the final teaching module be used by teachers in their classrooms with a minimum amount of disruption. This would mean that those using the erroneous addition strategy would not necessarily be isolated from their peers and so the teaching module would have to be used with entire classes. The next trial therefore took place with children who were in their normal class groups. It involved the use of the calculator to

(i) change fractional measurements to decimals
(ii) carry out all multiplication operations
(iii) find the scale factor.

The researcher taught a series of lessons (approx. 200 minutes) to classes of different ages and ability in the same school. The classes had been streamed according to ability and it was soon apparent that (i) above was a stumbling block for two groups as they did not connect 3 ÷ 4 with 3/4 and were convinced that division was commutative. Therefore the lessons they received were concerned with these aspects and the topic of enlargement was confined to doubling dimensions.

With two classes it was possible to pursue the teaching programme, the results are quoted below. The tests given to the children contained four items on which the incorrect addition strategy had been shown to be commonly used.

(1) A class of 15 year olds regarded as 'average' by their teacher (n=14)

Figure 1. Addition Strategy Type Answers.
Delayed Post Test (After 8 weeks)

pre-test	0	1	2	3	4
0	3				
1	1				
2	4		1		
3		2			
4	1	2			

Figure 2. Correct Answers

Delayed Post-test (after 8 weeks)

pre-test	0	1	2	3	4
0			2	2	2
1	1	1			1
2					3
3					
4				1	1

(2) A class of 12-13 year olds regarded as 'bright' by their teacher (n=15)

Figure 3. Addition Strategy Type Answers
Delayed Post Test (after 8 weeks)

pre-test	0	1	2	3	4
0	6				
1	2				
2	4	1			
3	1		1		
4					

Figure 4. Correct Answers
Delayed Post-Test (After 8 weeks)

		0	1	2	3	4
	0			1		
	1		1		2	
pre-test	2			3	1	2
	3			1	1	
	4					3

It can be seen that although the incorrect strategy can be eradicated; complete success in solving these ratio and proportion questions is not achieved. We do however know from the interviews, that without specific intervention (i.e. when the children simply attended their normal mathematics classes) the incorrect strategy is very persistent.

The module is now being taught by mathematics teachers in their own classrooms and it is envisaged that further changes will be needed after the teachers have reported their use of the material.

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The role of rough estimation in learning ratio and proportion

- an exploratory research

Summary

The stumbling block for learning ratio reported by many researchers is wrong additive reasoning. It is our main concern to supersede or even prevent this attitude. The place of ratio and proportion in the total curriculum is an important factor. Visualising models and patterns from the child's perceptive world can play an important part. In the present exposition the part played by estimination is stressed.

1 The main problems of learning ratio and proportion

1.1 The place of ratio and proportion in the curriculum

Ratio and proportion are related with scales and with equivalence of fractions. This relation as taught in the textbooks is weak, because of the abstractness and the advanced algorithmic character of these subjects in the curriculum. This relation should be strengthened.

1.2 The relation with the child's perceptive reality

Children acquire their first geometrical notions by organising their visual world, where similarity plays an important part and consequently ratio, proportion and proportionality (cf. Bryant, 1974; Freudenthal, 1978;Van denBrink en Streefland, 1979) However: '... the influence of all this experience had been neglected by theory and by research.' (Fuson, 1978, p.245). Children's perceptive reality is indeed a wealthy source for ratio and proportion (cf. Freudenthal e.a. (eds.), 1976). This fact should be used in teaching ratio and proportion.

1.3 The part played by visual models and patterns in the process of algorithmisation of ratio

Examples of tools to be used are: the multiply scaled numberline and ratiotables (cf. Streefland, 1982). The necessity to support the process of algorithmising ratio will become obvious in the followin section.

1.4 Wrong additive reasoning, explained and fought

Example: R has determined how many beads (of each colour and together) he needs to make a 12 times repeated pattern of 2 white, 5 black and 3 spotted beads on a string.
E.: how many do you need of each colour to have the pattern 15 times repeated?
R.: 15 patterns is 3 more than 12. Subsequently he adds 30 to the numbers of each colour, thus:

PATTERNS	$12 \xrightarrow{+3} 15$
white	$24 \xrightarrow{+30} 54$
black	$60 \xrightarrow{+30} 90$
spotted	$36 \xrightarrow{+30} 66$

What is the cause of this behavior? Many hypotheses are possible. The action itself has an additive character which conflicts with the multiplicativity of the structure (cf. Desjardins and Hétu, 1974). Or: insight is suppressed by the complexity of the arithmetical operations (cf. Karplus a.o., 1981). In difficult arithmetic children look for a way out, following intuitive preferences, as appears from the attempts of 'building up an answer' (cf. Hart,1981). This is a support to Karplus'-view.

Rough estimation might be a powerful weapon in the struggle against wrong additive reasoning.

1.5 Rough estimation

Little is known about estimation (cf. Trimble, 1973; Bestgen a.o., 1980; Schoen, a.o., 1981; Trafton, 1978). Trafton mentioned measurement as a field where estimation applies. There are some Dutch publications which related estimation and ratio (cf. De Jong, 1982). Kühnel (1925) too made this connection.

With respect to the function of estimation in computation, Trafton argued (1978, p.203):

> *'The use of a reference point suggests excercises that promote reflective thinking...'*. (cf. Teule-Sensacq and Vinrich, 1982, too).

Such a reference point - in Trafton's case an extra number added to the data in a given problem - will give support and direction to the act of estimation.

Two Dutch authors of schoolbooks once stated:

> *'Estimation helps to anticipate the train of thought to be followed.'*

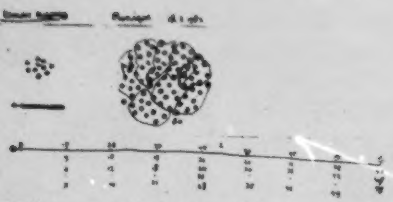
Moreover rough estimation will help the pupils to shift from a qualitative to a quantitative approach towards problems (cf. Van den Brink and Streefland, 1979).

1.6 Research conditions

Eleven third-graders (8 - 9 years) were confronted with five problems. The result of the third problem will be discussed in detail, if necessary compared with or completed by the remaining results. The problem chosen to be discussed is exemplary. All the problems had a strong visual component and in all of them measurement was involved. In the solving process one developed multiply-scaled numberlines and ratiotables. The children to be interviewed knew the multiplication tables, had some experience in counting large quantities and the distribution of quantities. Solving the problems included: understanding estimation being able to estimate understanding measurement, being able to measure with natural units, counting large quantities, understanding repeated addition, applying multiplication as a shortcut of repeated addition etc.

2 Some results

2.1 Description of the third problem



IIIa) Mary threads a small number of beads on string 1. She wants to thread a larger quantity on string 2.

The interviewer asks: 'Estimate the length of string 2 and draw it.

IIIb) After a) has been carried out the interviewer asks to check the estimate.

IIIc) The numbers are changed, multiply-scaled numberlines are established.

The interviewer asks: 'Suppose the beads are twice as thick, how many....'
'Suppose ... beads on string 1, how many on 2 ?'
'Suppose ... beads on string 2, how many on 1 ?'

2.2 Results of this experiment, supplemented and compared with the other results

2.2.1 Estimating, various methods (problem IIIa))

- 'Just' estimating and drawing the length of the string on this basis (2 pupils). One pupil tried to count the beads quickly.

Protocol:

R: It might be more than 100. I think 110.
E: I asked how long.
R: Well, but I estimate first and then I draw.

- Purely qualitative estimating: '... there are a lot of ... thus...' and drawing accordingly (1 pupil).
- Estimating the larger number globally and drawing the large thread with that of 10 beads as unit length (3 pupils); direct translation of the global number into the length of the string (2 pupils).
- Estimating by comparing with the group of 10 beads as a measure (2 pupils).
- Estimating by irrelevant criteria like: 'there are twice as much' (1 pupil).

Note: the third problem differed from the previous ones by the greater difficulty and complexity of the estimation procedure.

2.2.2 Checking; various methods (problem IIIb))

- *Counting the larger number*
  - starting with counting the larger number and then adapting the length of the string to the acquired result (9 pupils);
  - the strange converse method of measuring the drawn, estimated string by the unit measure of 10 beads in order to find the number of beads and subsequently

counting the larger number of beads and adapting the string accordingly (2 pupils). Counting the larger number was performed by making groups of ten beads (3 pupils); grouping after unsuccessful attempts of counting individually (3 pupils) and individual counting by one-to-one connections (5 pupils). The transition from estimating to checking elicited reflective behaviour:

Ex. Protocol (after the estimate):

E: *What are you going to do?*
I: *I do not know exactly.* (He starts drawing beads on the tread)
I: *Perhaps let us count these beads.* He starts counting and striking out the beads one by one and says:
I: *I got a better idea.* He counts the small number and tries to make groups of ten in the larger one. However groups of ten are not that easy visualised.
I: *I am going to use the table of five.* He indeed makes groups of five and provides them with a rank number.

- *The relation between the methods of estimating and checking*
  - agreement between both of them with regard to the arguments adduced (5 pupils)

Ex. Protocol (estimating cf. 2.1)

M seems to count. She draws the string.
E : *How did you do it?*
M : *I got 50. I counted by groups of ten and I made five pieces like that.* (checking)
M : *I see it is not 50.... first I thought....* (she counts the beads one by one).
E : (tries to confuse her) *Why did you count the beads?*
M : *If I know it precisely, I also know the length of the string.*
E : *Try it handily.*
M groups by 10 and produces the string correctly.

- *The part played by multiplication, when checking the estimate*
  - using the multiplication vocabulary while estimating (4 pupils). The same appears to be true of the great majority of the other pupils with respect to problem IIIc).

2.2.3 The part played by multiplication after change of numbers (problem IIIc)

The aim is producing ratios.

- *Applying multiplication and the associated vocabulary (11 pupils)*

  Not all pupils succeeded immediately. Some of them were inclined to fall back to:

- *Repeated addition (5 pupils)*

  This happened particularly when the diameter of the beads was doubled. Repeated addition was finally shortened to multiplication.

Ex. protocol

E: *Suppose the beads were twice as thick, how much would you need?*
J: *Then there are.... 40 beads.*
E: *How did you find out?*
J: *The beads are thicker so there are 5-5-5-5....* (pointing to the pieces of string).
E: *If there were six beads on the little piece of string.*
J: (applies the repeated addition as appears from her gestures): *48.*
E: *You used the pieces of string. Could you have predicted it?*
J: *No.*
E: *I think you could have. How many pieces?*

J: 8.
E: How many beads are threaded on a piece?
J: 6.
E: How many beads together?
J: 8 x 6.

- *The converse (given .. beads on the long string. How many on the short?)* – assigning an appropriate product in the table of 8 to the given number (10 pupils).
- Applying properties of the 'mapping' (... is 8 times aslong (many) as ...) which produces the terms of the ratio (cf. Kirsch, 1969; Vergnaud, 1982). During the activities which aim at producing classes of equivalent ratios the pupils will discover and apply the properties of the mapping involved more and more. Three of them did so in the third problem.

Ex.Protocol

E: If there were six beads on the small thread, how many on the long thread?
D: .... 48 I think.
E: What is my next question?
D: How I found it.
E: Tell.
D: I thought: from each 10 (initially on the small piece) take off 4! 4 x 8 = 32 thus 32 off.
E: Fine. And if the short piece had 4, how many on the long one?
D: First I thought again 6 off, that is 48. Just now I had also 48 and 32 more. Thus 32.

Note: when solving the fifth problem 9 pupils applied the mutual relationships of the numbers stored in a ratiotable. Discovery and (correct) application of additive relations happened most of the time. Taking into account the phenomenon of proportionality however, happened a few times too. Specially the operations of doubling and halving the pupils got aware of. What struck was the growing awareness of an increasing number of pupils of the computational possibilities involved in the properties of the mapping, while no special attention was payed to it.

2.2.4 Reflection on estimating

In all five problems the reflection on one's own activity was remarkable. In particular the third problen elicited reflection and certainly when the pupils had started with inefficient methods of estimating and counting (cf. protocol I). Reflective moments in the solving process could be ascertained with six pupils and made probable with two. Reflection occurred in various stages of the solving process. It was elicited by estimating or by checking and accordingly functioned in two ways: (1) estimating can elicit a global orientation on the problem set, which organises the problem domain and prestructures the solving procedure. Some pupils immediately grasped that the third problem involved two kinds of estimates (number → length), other discovered it, when checking. By means of reflection on the problem situation the operational structure of the problem can be uncovered. This means that estimating in this kind of ratio problems allows the children to anticipate the operational structure. This blocks the additive approach. The procedure that was applied in estimating as well as the estimate itself serve as 'reference points' as ment by Trafton (1978, p.208). It is an advantage that the 'reference points' rather than being imposed have been marked by the solver himself. (2) Gross conflicts with estimates or bad counting methods elicited reflection. Insufficient counting methods were modified and bad estimates were corrected. All cases were examples of 'directed' shifting in the sense of Freudenthal (1979). (Shifting from A to B while considering C). A final important remark refers to the fact that estimating allows the solver to start semi-qualitatively (semi-numerically). It is an advantage that more space is reserved in the conscious mind to uncover - as mentioned before - the operational structure of the problem.

3 Conclusion

Estimation proved to be a powerful tool in the solving process of certain types of ratio and proportion problems. Reflective thinking might be both provoked and developed by the demand of estimating. Other demands have to be taken into account too, like the solid anchorage of the subject matter in the curriculum and the development of tools to support the process of algorithmisation of ratio. In conclusion the provisional character of the results has to be emphasised. They should be considered as the first stage in the research of the long term learning process aimed at mastering the concept of ratio and proportion at an operational level.

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EVALUATION FORMATIVE DES NIVEAUX DE MAITRISE DES CONCEPTS DE PROPORTION ET DE POURCENTAGE.

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This study is about of the connexions between the procedures and general stages of intellectual development in the school learning context. The author analyses the answers of 192 pupils to a test of the calculation of percentages. The results show three successive levels within the stage of formal operations : level 1 is characterized by a strategy founded on the notion of mutiplying ratio , level 2 is characterized by an algorithm concerning the coefficient of proportionality research, level 3 is characterized by the awareness of a specific property of proportion.

INTRODUCTION

La psychologie génétique d'aujourd'hui s'attache à la description des procédures et des stratégies qu'elle analyse intensivement. Ce qui la conduit à laisser à l'arrière plan les stades généraux du développement intellectuel. Pourtant, toute procédure trouve sa véritable signification à travers un certain état global d'équilibre ou déséquilibre qu'elle tend à révéler. Piaget a eu sans doute raison de retenir à la place du terme équilibre le terme " équilibration" qui souligne à juste titre le caractère dynamique de l'élaboration et de l'organisation des connaissances.

Dans le but de mieux comprendre les rapports entre les procédures et les stades généraux du développemnt intellectuel, nous avons entrepris des études sur la maîtrise de certains concepts mathématiques dont l'acquisition efficiente apparaît plus probable à certains stades du développement.
Cette maîtrise des concepts mathématiques est reconnaissable, à notre sens, grâce à une certaine forme de réversibilité, caractère essentielle, selon Piaget, d'un état d'équilibre à cractère opératoire.

Dans ce cadre théorique, une opération mentale est dite réversible, lorsque partant du résultat de cette opération, on peut trouver une opération inverse par rapport à la première et qui ramène aux données de cette première opération sans qu'elles aient été altérées.
Ainsi, dans cet exemple du calcul de 22 % de 450, le calcul donne 99. Mais si l'on formule la question de manière à demander ce que représente en % 99 par rapport à 450, le calcul donnera 22 . L'opération relative au calcul de pourcentage est donc une opération réversible. La réversibilité consistant à rechercher la valeur d'un coefficient de proportionnalité exprimé dans ce cas en termes de " pourcentage ".

Selon Noelting, G., plusieurs études(Lovell, Butterworth,1966 ; Lunzer et Pumphrey, 1966,Fishbein, Pampu et Manzât, 1970 ; Strauss 1977), ont confirmé que le concept de rapport et de proportion est acquis tardivemant au cours de l'adolescence.
L'étude de Hart,M.,K., et Kerslake,D. ..., rapporte que sur 2 257 enfants de 11 à 16 ans examinés en 1976, 20 seulement savaient écrire et utiliser correctement une équation de la forme : $\frac{a}{b} = \frac{c}{d}$.
Nous avons à notre tour, essayé de distinguer quelques étapes essentielles de la formation du concept de proportion.

Notre souci en tant que psychopédagogue, ne se situe pas au niveau de la recherche des facteurs explicatifs des différences entre les procédures uniquement; outre son intérêt pour la source de ces différences, le psychopédagogue s'intéresse également et je dirais surtout, à comprendre en quoi celles-ci peuvent rendre compte des progrès ultérieurs des individus qui les manifestent. Les questions qu'il se pose sont celles qui se rapportent aux adaptations successives corrélatives à l'évolution des différentes stratégies mises en oeuvre par les sujets au cours de l'apprentissage.
En d'autres termes, le problème fondamental de la psychopédagogie est, selon nous, celui de la régulation des apprentissages. Il s'agit de savoir comment aider le sujet à passer d'un type de procédure à un autre plus opérationnel que le premier.
Parallèlement à cette question, on peut se demander si tout changement de procédure correspond à un changement d'acquisitions et lesquelles ? Quels sont les facteurs qui permettent d'accélérer des tels passages ?

Cette étude traite de la différenciation des niveaux d'acquisition de la notion de proportion à partir des procédures qui s'y rapportent. Elle vous sera présentée en quatre points : 1°- Notion de proportion 2°- Approche de l'étude du concept de proportion 3°- Analyse et discussion des résultats 4°- Conclusion .

1.Composantes du concept de proportion

Nous considérons une proportion comme une égalité entre deux rapports pouvant être présentée sous la forme : $\frac{a}{b} = \frac{c}{d}$.
Certains auteurs (Freudenthal et Noelting notamment), ont distingué entre une proportion traduisant des rapports du système interne d'une part (within-state ratios) et une proportion traduisant des rapports du système externe d'autre part(between-state ratios). Dans le premier cas, l'analogie évoquée concerne des termes de même nature, comme par exemple dans la comparaison des unités de mesure de longueurs :
$\frac{1 \text{ Hm}}{1 \text{ Km}} = \frac{1 \text{ dm}}{1 \text{ m}}$.
Dans le deuxième cas, l'analogie évoquée, associe des termes de différente nature. comme dans l'exemple de la cinématique : $\frac{v}{1} = \frac{e}{t}$
Selon Noelting, G., cette distinction de deux types de rapport, proviendrait de deux sortes de processus cognitifs chez le sujet : (1°) Assimilation des éléments semblables avec des variations d'un élément suivant une qualité particulière. (2°) Relation entre différents éléments avec la construction d'un nouveau concept : aRb, cRd . (1)

Cette distinction des composantes de la proportion peut se faire du point de vue de la formation suivant le principe de la réversibilité des opérations mentales. Selon ce principe, une proportion peut être examinée non seulement sous sa forme $\frac{a}{b} = \frac{c}{d}$, mais aussi sous la forme $\frac{c}{d} = \frac{a}{b}$ .Cela signifie en pratique que les éléments de la proportion ayant joué le rôle d'inconnue peuvent jouer à un autre moment le rôle de donnée , ce qui permet d'inverser les opérations.
Cette distinction qui semble ne rien apporter de plus du point de vue mathématique, a cependant une certaine importance du point de vue psychologique. Elle permet,en effet, la prise de conscience des perturbations que pourrait engendrer la modification des données de la situation problématique.

2. Approche de l'étude de la formation du concept de proportion

L'intérêt d'une approche psychopédagogique de la formation des concepts par le biais des procédures, réside sans doute dans le fait de la possibilité de savoir , lorsque certaines conditions sont réunies, si l'usage des différents types

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1)NOELTING,G., The development of proportional reasoning and the ratio concept. in Educational Studies in Mathematics. Vol.II N° 3 AUG. 1980 , pp . 352 - 361 de procédures est induit par des différences dans les situations ou par des différences psychologiques entre les individus.

La présente recherche s'est attachée au premier aspect de ce problème : à savoir l'étude de l'influence des modalités de présentation d'un problème sur le traitement et les performances des sujets.
Pour cela, une épreuve comportant 12 problèmes relatifs au calcul des pourcentages a été mise au point et présentée sous deux formes équivalentes A et B contenant chacune 6 items.
Ces deux épreuves ont été conçues suivant la méthode de l'évaluation formative.
Par " évaluation formative ", nous entendons l'évaluation ou la mesure des acquisitions du processus de formation en vu de l'amélioration de ce dernier. Il s'agit d'établir à partir d'un certain nombre d'informations, l'état de connaissances du sujet du point de vue de l'acquisition d'une capacité spécifique. Il s'agit en fait de tenter de répondre aux questions de savoir:où en est le sujet du point de vue de l'acquisition d'une telle capacité ? Quels sont les schèmes d'actions dont il a pris conscience ? Quelles connaissances met-il en oeuvre sans en avoir conscience ? Comment réajuster son apprentissage pour lui permettre d'atteindre l'objectif optimal ?
Pour obtenir des données permettant de traiter chacune de ces questions, l'instrument d'évaluation nécessite l'élaboration des items touchant aux divers aspects de la formation du concept ou de la capacité qu'il s'agit d'étudier.

L'épreuve qui a permis d'obtenir les résultats qui sont présentés ici comporte quatre classes d'items :

La forme A est composée de 6 problèmes ayant trait à des situatiosn pratiques.
Ces items sont répartis en deux classes : la classe A_1 contient des items portant sur le calcul de valeur d'un pourcentage donné. Il s'agit des items numérotés 1,2,3 ? La classe A_2 est composée d'items portant sur la recherche d'un coefficient de proportionnalité exprimé en pourcentages
La forme B comporte également deux classes d'items B_1 et B_2, établies suivant le même critère que pour A. Mais tous les items de l'épreuve B ont pour unique donnée des nombres sans lien avec des situations pratiques.
Nous avons invité 192 élèves (des classes des collèges) de 11 à 16 ans à résoudre en deux séances, ces deux épreuves avec un intervalle de temps d'une semaine.

3. Analyse et discussion des résultats

Les résultats obtenus montrent d'une manière générale que 25,5 % des sujets (49 élèves), n'ont pas acquis la notion de proportion. Ces sujets n'ont pas su résoudre un seul item de l'ensemble des problèmes proposés. Leurs réponses sont faites d'additions élémentaires ou des divisions sans lien avec les opérations ou l'algorithme sollicités par la nature des relations existantes entre les données du problème.

Ce n'est qu'à partir de la classe de 3e que l'on trouve 8 élèves , soit 4% de notre échantillon, qui réussissent à l'épreuve dans sa totalité.

Quant aux sujets intermédiaires, ceux qui obtiennent un score peu élevé dans l'épreuve globale (2 ou 3 bonnes réponses) , ne maîtrisent que les items de la classe A_1 ; ceux qui ont obtenu un score élevé (entre 8 et 11 bonnes réponses), échouent en général aux items de la classe B_2. Les questions les plus difficiles apparaîssant comme étant celles qui portent sur la recherche du coefficient de proportionnalité et dont les données sont les nombres sans lien avec des situations pratiques.

Les sujets ayant obtenu un score moyen (entre 4 et 7 bonnes réponses) ne présentent pas de faiblesse particulière. Ils réussissent à quelques items de différentes classes.

Les réponses correctes et mauvaises à chaque item sont ainsi réparties :

	Items	Epreuve A			Epreuve B		
		R	W	NR	R	W	NR
Classe 1	1	104	74	14	110	70	12
	2	49	138	5	61	117	14
	3	63	112	17	62	111	19
Classe 2	4	36	144	11	34	126	32
	5	44	125	23	26	110	56
	6	42	118	32	31	105	56

R = bonnes réponses

W = Mauvaises réponses

NR : sans réponses

N = 192

Tableau 1 : Répartition des fréquences des réponses correctes et incorrectes par item.

Certains sujets présentent une sorte de dominance, c'est-à-dire qu'ils réussissent mieux aux items des classes A qu'à ceux des classes B.

Les sujets ont utilisé des stratégies très diverses. Parmi les plus fréquentes, on retrouve les deux types de procédures décrits par Noelting :- la procédure dite de la covariation dans laquelle le rapport fonctionne comme un opérateur. Le sujet ayant maîtrisé la notion de réduction des fractions . On obtient 4 fois cette réponse :

22 % de 750 c'est 0.22 . 750, 35 fois cette réponse : 22 % de 750 c'est 22.7,5 . D'autres élèves utilisent la division des termes et la comparaison des quotients par rapport à l'unité. Nous avons obtenu 58 fois cette réponse 22 % de 750 c'est (22. 750) : 100 = .

D'autres enfin, on mis en oeuvre une procédure plus élaborée : la mise en équation. La solution de l'équation étant fondée sur cette propriété de la proportion :
$\frac{a}{b} = \frac{c}{d}$ entraine a .d = b .c entraine $a = \frac{b.c}{d}$
Ce type de procédure est obtenu en général chez des élèves plus âgés.
Exemple de réponse de F.M.,(15ans) $\frac{224}{280} = \frac{x}{100}$, entraine 280 x = 224 . 100 , x = ..

Il semble que plus le sujet est confronté à des problèmes faisant appel à l'abstraction plus il a besoin d'outils opératoires qui lui permmettent d'échapper à la confusion perceptive. La recherche du coefficient de proportionalité n'est abordée correctement que par ceux qui maîtrisent la propriété évoquée ci-haut.

4. Conclusion

L'analyse détaillée des résultats semble reflèter la succession des ... opérations suivantes : la maîtrise des rapports multiplicatifs entre les nombres permet à certains sujets de chercher la portion par récurrence, par des divisions successives. Ceci serait la caractéristique d' un premier niveau se rapportant à des sujets qui n'ont pas encore donné à l'égalité le satut d'un invariant. La découverte du coefficient de proportionnalité les aide à y accéder à un deuxième niveau. Mais sans que le sujet puisse traiter toutes les combénaisons possibles des rapports avec les mêmes termes. Ce que l'on obtiendrait par exemple en inversant numérateurs et dénominateurs. L'étude des propriétés de la proportion l'élèverait alors à un niveau de haute maîtrise de la proportion.

Mais toute la lumière sur cette question ne peut être faite que par une approche des thèmes mathématiques plus larges dont la notion de proportion fait partie. Pour notre part, nous partageons cette conclusion de Hart, K., qui écrit : " Children who are not at a level suitable to the understanding of $\frac{a}{b} = \frac{c}{d}$ will just forget the formula " (2)

Annexe : Epreuves de calcul des pourcentages

Par manque de place, nous donnons uniquement la forme B .
la forme A étant équivalente à la forme B avec la différence provenant du fait que les problèmes de la forme A font référence à des situations pratiques comme nous l'avons précié dans le texte.

Classe d'items B_1

1. 22 % de 980 c'est ....
2. 20 450 réduit de 6 % devient ...
3. 45 600 augmenté de 35 % devient ...

Classe d'items B_2

4. Si 3 550 devient 3 834, on l'a augmenté de ....
5. 255 c'est ..... % de 340
6. Si 210 devient 199,5 on l'a réduit de ...

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G. METHODS OF TEACHING

TEACHING THEORIES IN MATHEMATICS

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This paper summarises research on teaching methods for certain key mathematical topics and discusses the results in relation to theories of teaching and learning. It is a sequel to our paper for the 4th PME Conference (Bell, O'Brien and Shiu 1980). Further reports of this work are available (Bell and Shiu 1981, Shiu and Bell 1981).

We assume that the aim of mathematical education is to develop pupils' capacity for mathematical activity, and to help them to acquire the knowledge and skill structures which this demands. Mathematical activity consists of the abstraction of relational structure from a given situation or context, its representation by symbols or diagrams, the making of transformations of the symbolic representation in order to expose some new detail of the structure, and the interpretation of this new item in terms of the original situation. This is the characteristic activity of applied mathematics; an example is the solution of verbal problems in arithmetic. Pure mathematics performs the same kinds of operation on the structures themselves with a view to exposing new properties of them; a particular form of this is the development of rules for transforming the symbol-systems in ways consonant with the properties of the represented structures. This is the orientation which guides our choice of curriculum tasks for study.

The design of mathematical curricula has traditionally begun with the choice of those structures (and algorithms) which seemed useful and accessible; these have been taught along with applications to various contexts, the choice of the contexts not being regarded as important. Recently, awareness has grown of the extent to which the structures actually used by pupils depend on the context (Wason and Shapiro 1971), and on the level of abstraction of the elements involved, e.g. small integers, large integers, decimals (Hart 1981). This demands a reappraisal of this tradition; context, type of element and other psychologically relevant aspects need to be specified in any meaningful description of a curriculum.

TEACHING THEORIES

Theories of learning and teaching need to cover three questions:

(1) What are the basic mechanisms of learning?
(2) How do these relate to different types of acquisition, and to different types of learner?
(3) What laws govern the degneration or forgotting of once-learnt material?

Two theories which might thus explain mathematical learning are those of Gagné and Piaget. In the Gagné theory an analysis of mistakes leads to the establishment of a hierarchy of the subskills needed to build up to the desired skill. Each learning step consists of the combination of two or more subskills to form a more complex skill. Whether this is accomplished with the aid of verbal instruction or as an act of discovery is not specified; nor whether the reinforcement is extrinsic, by feedback of correctness, or by the intrinsic reward of insight. However, Gagné does distinguish a number of different types of acquisition, from the learning of names for already-known concepts or actions (signal learning), through concept learning, to the learning of cognitive strategies; and he specifies the conditions which facilitate each type of learning.

For Piaget the basic learning process involves assimilation and accommodation; new material is assimilated to existing cognitive structure, but if the attempt involves too great a 'cognitive conflict' this is resolved by a change in the cognitive structure: an accommodation. Thus Gagné's theory is an 'accretion' theory; it assumes that new knowledge is added on, with suitable linking, to existing knowledge, while Piaget's is a 'restructuring' theory; it assumes a pre-existing mode of response to any new input, which is then modified. Piaget does not distinguish different types of *material*, but does specify different modes of learning, in terms of the reasoning patterns available at different stages of intellectual development.

Our teaching experiments have been conducted in a Piagetian framework of ideas, and many of the findings are clearly only interpretable within this theory. Some of its principles, such as that of cognitive conflict, will, however, be the subject of experimental test.

The experiments have had two types of outcome. First, they have had implications regarding the structure of the conceptual systems possessed by our pupils and brought to bear on the topics and tasks offered. Secondly, there are indications regarding the teaching principles which seem to be effective.

The topics studied were (a) directed numbers, with addition and subtraction, (b) decimal numbers, with multiplication, and (c) additive problems involving both particular and generalised quantities. Space only allows the first topic to be discussed in this paper.

DIRECTED NUMBER

Existing school courses tend to begin by illustrating the concept of negative number in a number of contexts - temperature, co-ordinates, money, heights relative to sea level - then to introduce the number line, and to define the operation of addition with reference to the line only. Subtraction is occasionally defined as a displacement on the line, but more commonly as the addition of the additive inverse, or the 'opposite'. (At least one course defines the first minus sign in $3 - {}^{-}4$ as 'face west' and the second as 'walk backwards', so that there is no *operation* sign at all.) In subsequent work it is rare to find any discussion of the *meaning* of addition or subtraction of directed numbers in relation to any context, though calculations such as $(3 - {}^{-}4)/({}^{-}5 - 1)$ are performed by rule, or perhaps by visualising the displacements. There is a case for regarding the study of bank balances and transactions, relative heights, fast and slow clocks and combinations of additive and subtractive operators as worthy of a place in the curriculum, alongside the co-ordinate plane, because of the importance of these contexts, as well as for their value in providing for a fuller conceptualisation of operations on directed numbers.

THE CONCEPTUAL FIELD OF DIRECTED NUMBERS

Interviews were conducted for the pre and post tests for each teaching experiment; the teaching experiments were with two pairs of pupils and one individual. The following aspects were tested:

(1) addition and subtraction of pairs of integers ($3 - 10$, $234 - 589$, ${}^{-}5 + {}^{-}9$, ... $7 - {}^{-}2$)

(2) ordering ($^-3 \gtrless {}^-7$)

(3) rearrangements ($5 - 12 + 8 - 3 \overset{?}{=} 3 - 8 + 12 - 5$)

(4) understanding of 'change side, change sign' in equations (if $10 - x = 7x + 6$, complete $12 - x = ...$)

(5) testing and explaining generalisations ($^-11 + 3 = 3 + {}^-11$, smaller - bigger = negative for integers)

(6) applications (does FINAL STATE - START STATE = INCREASE work for negative bank balances; what move takes ($^-3$, $^-2$) to ($^-5$, 4)?

Following pilot testing, the following levels were defined; and subsequent testing of a group of 25 15 year olds gave the results shown in the table:

LEVEL 0 - either has no knowledge of negative quantities OR cannot integrate knowledge of negative numbers with known properties of positive numbers.

LEVEL 1 - recognises the existence of negative numbers and can order any pair of integers.

LEVEL 2 - can reliably add any pair of integers of small magnitude.

LEVEL 3 - can reliably subtract any pair of integers of small magnitude.

LEVEL 4 - can order, add and subtract integers and use these abilities to test generalisations concerning the properties of directed numbers.

LEVEL	Unclassified	0	1	2	3	4	Total
NO. OF PUPILS	1	3	1	10	8	2	25

(A subsequent written test of 47 pupils gave similar results.)

Thus in this interview sample of 25, 20 (80%) were successful at addition and ordering, 10 (40%) at subtraction, and 2 (8%) at applications and the testing of generalisations.

Addition was in general performed meaningfully, with reference to the number line or to ideas of 'quantities less than zero'. (For example, $^-5 + {}^-9$ was seen as the addition of two quantities of the same kind.) For subtraction, most pupils had no such conceptualisation, but worked from rules such as 'subtract is go to the left' or 'two minuses make a plus'. These rules were subject to extensive degeneration; for example, $^-9 - {}^-2 = {}^+11$ because "minusing two negatives equals a positive", and $7 - {}^-2 = 5$ by "negative is to the left", and "subtract is go to the left" combined with reinforcement, not reversal. The expression $5 - 12 + 8 - 3$ tended to be seen as two pairs of numbers subtracted, e.g. $(5 - 12) + (8 - 3)$ and the correctness of rearrangements judged by whether these pairs remained intact (or reversed). The difficulty with the application to bank balances lay in overcoming the reversals of the time order, initial state - increase - final state.

These observations from the interview tests provide information about the conceptual field of directed numbers in the pupils' cognitive structure. The main needs seemed to be the enrichment of the conceptualisation of subtraction and the meaningful appreciation of the rule that subtraction is equivalent to adding the opposite. The teaching, aiming at these, provided further information about the interaction between the chosen situations and the pupils' framework of ideas. Its success also increased our confidence in the diagnosis.

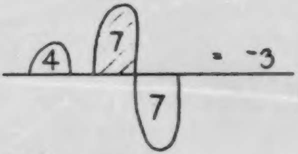
A money situation involving a person's state of account on each morning of a week, and the transactions taking place each afternoon, was formalised, after initial discussion, as shown in the diagram below. (Easier and harder versions were also used.)

	Mon		Tues		Wed		Thurs		Fri
Had	-3		☐		-9		☐		$^+4$
Gained/Spent		$^-4$		☐		☐		$^-7$	

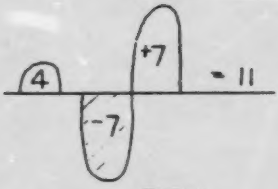
Filling these boxes requires three types of operation, (1) addition, of the form Before + Change = After , as for the Tuesday morning state, $^-3 + {}^-4 = {}^-7$; (2) subtraction of the form After - Before = Change, as for obtaining the Thursday state, $^+4 - {}^-7 = {}^-11$. The first of these two subtractions is commonly avoided by thinking of 'making up', $^-7 + ? = {}^-9$; and its relation to 'adding the opposite' is not at all obvious. The second may be suitably avoided, but in this case the addition of the reversed $^-7$ to the $^+4$ is fairly natural. In either case, to expose the directed number operation, the pupils need to be asked to solve similar problems with harder numbers or, for example, to write a formula by which a machine could check each calculation.

An exercise involving the representation of one of the previous money situations by points and arrows on a number line proved quite difficult and apparently valuable, developing the conceptual discrimination between time order Monday-Friday and order on the line representing level of current balance.

Two situations were developed specifically to embody the 'take-away' aspect of subtraction for directed number (since this had been shown to be a persistent idea, leading to the $^-11 - 6 = {}^-5$ error). The first of these was 'Heaps and Holes'. In this model, 4 - 7 requires starting with a heap of earth size 4, and needing to remove a heap of size 7 - this requires digging a hole, and a hole size 3 is left when the heap of 7 has been taken away. Diagrams as shown were used, but we encouraged thinking of the heaps and holes as far as possible rather than relying only on the diagrams.

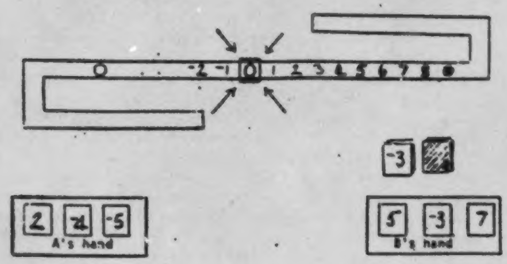


$4 - 7 = 4 + {}^-7$
$= {}^-3$



$4 - {}^-7 = 4 + {}^+7$
$= 11$

The game 'Directed Number Rummy' is illustrated below:



Players are dealt 3 cards at the start and take up position according to the sum of their cards. Each turn consists of picking up either the exposed or the unexposed card, and throwing one card down onto the 'exposed' pack. Thus bringing a card into the hand is adding, and throwing a card away is subtracting, and it becomes very clear that, for example, throwing away a negative card takes you forward on the number track.

TEACHING PRINCIPLES

We have adopted a Piagetian framework of ideas for this work; that is, we have sought first to ascertain what is the existing cognitive structure possessed by the pupils in relation to the tasks in question, and to attempt to expand and refine this structure by posing tasks which lead to cognitive conflict. Thus we have chosen contexts which relate as closely as possible to the pupils' existing experience, and have probed and developed the awareness of the relational aspects of these contexts, using the pupils' own modes of reasoning. This is, in our view, the essence of mathematical activity. This adoption of the pupils' own modes of reasoning conflicts with the usual view that the teaching of mathematics can proceed by the deductive derivation of mathematical structures. This false view underlies the assumption that it is desirable to justify, deductively, new mathematical principles as they are introduced. Examples of this feature of teaching method which impinged on our work were (1) the justifications offered of the law for placing the decimal point in a product (by recourse to the corresponding fraction laws), and (2) the introduction of compositions of directed numbers initially as newly-invented operations, subsequently shown to correspond to adding and subtracting for positive numbers. These two approaches both figured in 'control' teaching methods in our experiments; both were clearly out of tune with the mode of thought of the pupils, and no trace of either appeared to remain in their consciousness, at the time of the subsequent interviews.

The second teaching principle adopted concerns transfer. It is Wason, rather than Piaget, who has drawn our attention most strongly to the high degree of dependence on familiarity of context of both the difficulty of reasoning tasks and of the modes of solution adopted. Existing mathematics courses tend to offer application-problems as if to make the mathematics more interesting and to demonstrate its relevance to daily life. They do not take seriously the needs for familiarisation with all the relational aspects of the context itself, and for the conscious study of the way in which the mathematical structures in question fit most naturally to that particular context. (To give one example, the application of proportion to linear situations and to enlargement in two dimensions, as in the Piaget/CSMS eels and K tasks, present quite different problems). Transfer is a much more difficult and serious task than has been generally supposed. Examples from our work include the failure of many pupils to transfer directed number knowledge to the task of rearranging $5 - 12 + 8 - 3$, seeing this as composed of binary compositions, and the general failure to transfer the number knowledge to applications. The implications are (1) that the contexts which are or will be important in the pupils' experience need careful choice and substantial treatment, (2) that translation between contexts and the recognition of isomorphisms between contexts need to be used more fully as learning exercises. Our experiments showed that this was quite feasible and that it appeared to produce significant learning. A negative implication is to suggest caution in offering illustrations or embodiments of a principle to a pupil, particularly when the difficulty arises from a symbolic task; the pupil may understand the illustration but not see the connection between it and the task in hand. An example of this occurred when the decimal point principle for products was illustrated by recourse to lengths and areas on a square grid.

A third teaching principle follows from the Piagetian view of the pupil as actively involved in learning, as opposed to being conditioned to produce automatic responses. We found that important general concepts such as state and change, or number-sign and operation sign, could be recognised quite easily by pupils, and that this enriched and improved their ability to operate successfully.

Fourthly, rules which relate to and unite a body of experience can serve the same function as keystones in the structure of knowledge as the major theorems in a mathematical theory; they aid the memorisation of the whole structure. 'Subtraction is the same as adding the opposite' can serve in this way, provided it is well linked with experiences in a number of contexts. But 'two minuses make a plus' is not intrinsically meaningful; it refers to symbolic manipulations and not to the underlying concepts. It showed in our experiments to be very susceptible to garbling. Other rules similarly hazardous were those referring to right or left; the binary confusion is almost inescapable.

The last teaching principle needing mention is that of cognitive conflict, with the exposure of misconceptions for discussion. This also relates to the principle of pupil participation and awareness, and to 'general concepts'. This principle is almost self evident. A pre-existing faulty procedure cannot be replaced by a correct one unless some signal is introduced indicating its incorrectness. For example, it appeared distinctly helpful for pupils to recognise, that $^{-}11 - 6$ was ⑪ - ⑥ or negative - positive, and that this was the difficult case. This warning signal made them pause to think more carefully and to get it right.

These principles all appeared to be supported by the observations and the results of our teaching experiments; the results of the interviews also supported a Piagetian interpretation of learning and of the degeneration of knowledge structures. Further experiments planned will seek to test the effectiveness of conflict and the exposure of misconceptions as against 'positive-only' teaching.

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Clarity about the meaning of the equal-sign by "Playing at Balancing".

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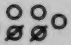
At the outset

In our work at the Department of Special Education of the University of Nijmegen we are especially concerned with arithmetic disabilities. For many years we have worked along Gal'perin-like lines, starting upon a material basis and developing the children's mental actions by step-by-step abbreviation and internalization of material actions.

One of the first questions that arise is: what material actions are most suited as the basis of beginning artihmetic?

Starting on the basis of whole-part-relationship has always been practically universally accepted.

In the case of addition this basis seems to work: 4 and 3, 7 together, in a formula: $4+3=7$. "Seems" to work, for if we put the sum differently, like $4+.=7$, children often find $\boxed{11}$ to be the solution, taking the 4, the 7 and the +-signs as the only relevant data, the equal-sign as negligible, as a sort of signal standing for "find the solution". This is demonstrated most clearly in the sum $2\boxed{?}1+1$, solved as $\boxed{4}$.

Subtracting is even harder. Many children keep on naming this situation  as "3-2", not "conserving" the whole when split up in parts. Put in a different order we meet the same "signal"-idea with $.-2=3$, leading to $\boxed{1}$ as the solution. Sometimes "right" by chance, like in $8-.=2$, solution $\boxed{6}$, the value of which is only traceable by asking the children to actually perform the material actions. The trouble with the equal-sign is the extreme vagueness of its meaning.

To a new material basis: "comparing"

The idea occurred to us to ch-ose a new material basis, the balance, taking advantage of the children's experiences with the see-saw.

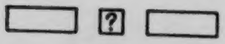
Inspired by the Russian psychologist Davydov we developed a programme on comparing. Our video-production "Playing at Balancing" provides the content of this programme in detail.

The following outlines the essential points.

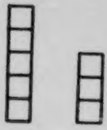
a. Before comparing numbers we make the children compare things on many other attributes like length, height, weight, volume, area etc., using the symbols =, > and < , so that they find out that things can be equal in height but nevertheless differ in volume:

Thus they come to understand that before comparing things there should be an agreement as to which attribute is to be considered. This leads to overcoming the so called "Piaget-faults" of e.g. confusing the height of liquid with the volume or - once the idea of number has been introduced - confusing the length of a row with the number of the elements in it. Essentially this provides the children with a clear view of what "number" is about and - equally important - what "number" is not about.

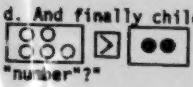
b. After they have mastered comparing things children learn the operations needed for the transitions from equal to unequal and v.v.
Given these two slips (to be compared as for their length) which one has to be shortened if we put the <-sign?



c. Next differences and similarities are to be determined:
Make blue what these have in common and make red where they differ:



d. And finally children will succeed in making (un)equal.



"Which operation(s) lead(s) to an equality concerning "number"?"

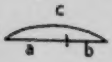
What are we aiming at?

We have three essential points in view:

1. Have children look at things analytically instead of roughly.
2. The comparison-programme leads to a visualised and general concept that can be used as a mental crutch for all possible transitions from equality to unequality and back again.

So these two simple lines: A B

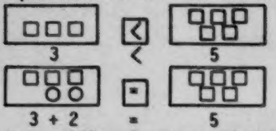
- stand for all measurable features on which things can be compared:
A is of greater value than B
A goes faster than B, etc.
- help us to find the operations needed to attain equality, greater inequality or opposite inequality.
- can be dynamically interpreted as in



where all imaginable values can be given to a and b.

3. Within this general framework of playing at balancing in which pupils become familiar with the comparison-laws in relation to all imaginable and

measurable features, we place the first sums, real comparisons, leading to equations as well as to inequations, e.g.:



A specific advantage of this procedure is the immediate and complete connection between material action and its symbolization. This stands in contrast to the whole-part-scheme as a starting point, which itself has two inherent difficulties:

- the signs for inequality are not used; exclusively equal does not mean much.
- the formulae don't reflect the actions as they do in comparisons:

000~~00~~		00000 > 000
5-2=3	versus	5 > 3
		000~~00~~ = 000
		5-2 = 3

Up to this point things are going smoothly even with slow learners. Problems arise when children who have been taught the comparison-model in our department (where they come for special help twice a week) are at school confronted with the only current model of whole-part-relations, which is doubly unfortunate:

1. For completely different situations:

000XX	and	000XX = □□□□□
3+2=5		3+2 = 5

the very same equal-sign is being used. Although it is true that "make together" versus "make together as many as" might be interchangeable at an abstract level, they are not and will never be alike at the material level. So we would strongly advocate using a different sign for "make together" - where comparing and together with this the signs > and < are not even as much as introduced - e.g. an arrow: 000 XX symbolized as 3+2 → 5.

2. In elementary schools the principles of whole-part-relations remain undiscussed. Casual examples of them that do occur, require a general framework, explicitly taught to children, based on material whole-part-relations referring to the length, the weight, the volume and other attributes of objects as well as to numbers and leading to a visualized concept like $\overbrace{a+b}^{c}$, from which all specific exampels can be inferred.*

In our view children need both models, which are different and indispensable for the whole field of arithmetic and problem solving.

* We have developed a programme on this subject based upon the ideas of G.G. Mikulina, as described in "Psychologische mogelijkheden voor het oplossen van opgaven met lettergegevens" (interne publicatie I.P.A.W., R.U. Utrecht).

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On Teaching the Comparison of Metric Measurements in a Vocational School

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I. Introduction

What is the most effective way to teach students mathematical topics that they have seen before, but have failed to master? Dienes and Goldring (1971), Kieren (1971), Whitcraft (1981), and many others advocate that the material should be presented through an alternate (non-standard) setting which is concrete and relevant to the student; e.g., vis-a-vis games or group activities which have as their underlying bases, the goals of the lesson. This approach, however, which seems to be so logical in theory, does not, for a variety of reasons, work well in practice. Bright, Harvey and Wheeler (1977,1980) and Friedman (1978) have shown that the effectiveness of the activity approach, in general, is questionable, and particularly so when focused on students in compensatory programs.

Authors of textbooks for students in compensatory programs are uniformly consistent in their approach to the problem. The usual sequence of material is presented, but at a slower pace, in simpler language and often times in larger print on pages with ample margins, which are used for jokes, illustrations and other miscellanea. Much attention has been paid to packaging, but students still seem to be unable to handle the material, regardless of the cosmetics surrounding the presentation (Eisenberg, 1981). The problem remains: how should students (especially older students, ages 14 and up) who are in compensatory mathematics programs be taught?

Presented here are the results of an experiment which used three different methods of instruction to teach 9th and 10th grade students how to determine the larger of two measurements. The measurements were stated with and without units. Each student in the experiment had been confronted with this task many times before.

II. Population

The students in the experiment were socially disadvantaged 9th and 10th graders studying in a vocational school in Beer Sheva. These students, at the end of 10th grade continue their formal education by studying only one day per week in the school, the remaining time being spent working as an apprentice in a local industry.

In a previous study by Hoz and Gorodetsky (1980) it was shown that 80% of the students in this school (ages 15-17), when asked to choose the larger between 0.38 and 0.7, chose the former. The students argued that it was larger because it had more digits. Israeli students first encounter decimal fractions like these in the sixth grade, and they are studied each year thereafter.

III. Experiment

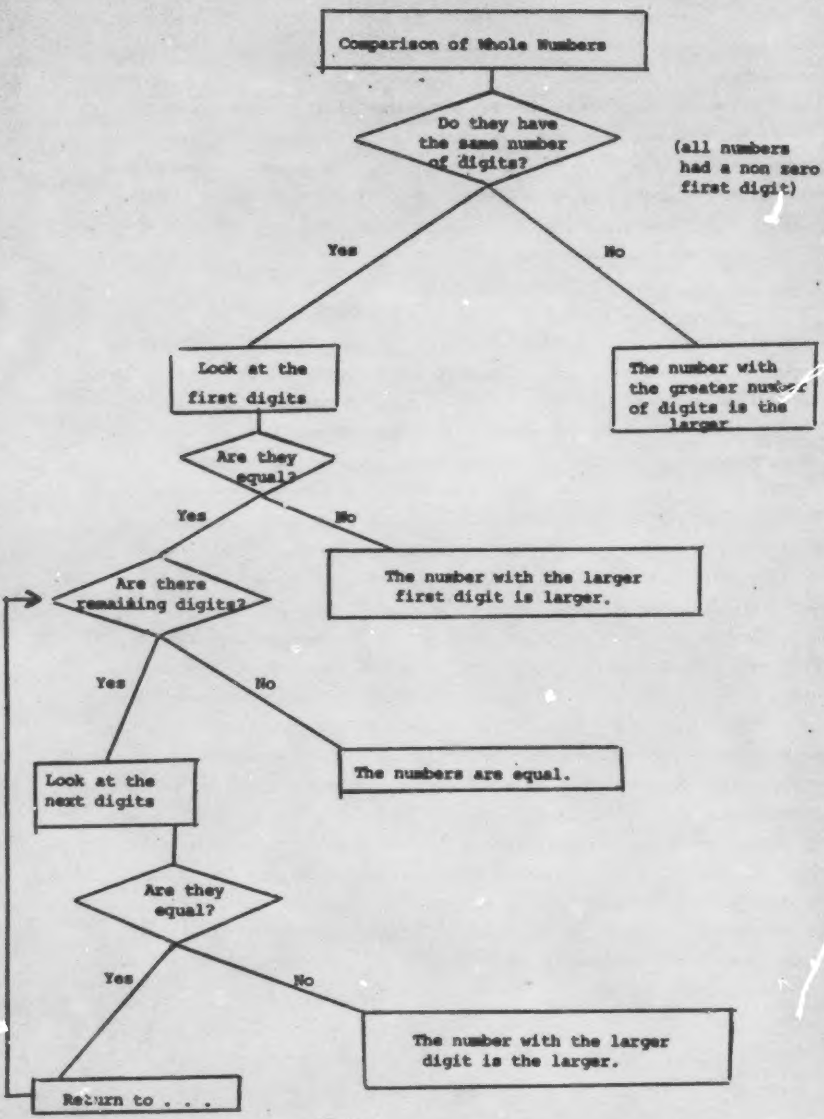
The students were divided into three groups. These groups were determined by a pretest given to the students at the start of the experiment, as well as IQ and psychometric scores on file at the school.

Each of the groups was taught how to determine the larger of two measurements, stated with and without units (e.g. which is larger, 2.5 m or 251 cm?). In the instructional program, only units of length were used. Units of time and weight comparisons were used as transfer items on the post-test. The post-test also served as the pre-test. Exercise sheets dealing with each comparison type, and combinations of them, were written. All exercises were completed during the regular class period.

Group I students were taught in a manner the teacher thought most appropriate to achieve the goal. Much time was spent on place value and the underlying idea of the metric system. The exercise sheets were used in the intended order, although the teacher was not compelled to use them at all. The students made practice comparisons on whole numbers, decimal fractions, simple fractions, decimal versus simple fractions and measurements with units, respectively.

Group II students were taught vis-a-vis an algorithmic flow chart which stated rules that should be used to make the comparisons. Each student had a copy of the chart in front of him and the teacher used the chart and the accompanying exercises explaining the lesson. The charts progressed in the aforementioned manner. Fig. 1 presents the algorithm used for determining the larger of two whole numbers.

Fig. 1


Comparison of Whole Numbers
Do they have the same number of digits?
(all numbers had a non zero first digit)
Yes
No
Look at the first digits
The number with the greater number of digits is the larger
Are they equal?
Yes
No
Are there remaining digits?
The number with the larger first digit is larger.
Yes
No
Look at the next digits
The numbers are equal.
Are they equal?
Yes
No
The number with the larger digit is the larger.
Return to . . .

Group III worked solely from a problem solving approach. The students were given, from the start of the experiment, exercises in comparing measurements with units. (e.g. which is larger ½ m or 0.4 m?). They were taught to analyze each problem individually, the idea being that the skills needed for the prerequisite comparisons (of whole numbers, decimal fractions, etc.) would also be learned vis-a-vis the context of the larger problem (see e.g. Shulman, 1968).

The experiment was conducted twice, both times in the same school. On the first run through, one experienced teacher taught all of the three groups. On the second run through, three veteran teachers were used, each being assigned a specific group. Each run of the experiment took two weeks (for a total of six hours of instruction) and each lesson was observed by at least one of the experimenters.

The post-test had 13 categories of questions similar to the following: Write the appropriate symbol (<,=,>) between the following two measurements: 0.20 m ____ 0.2 m. Similar questions were asked on weight and time measurements as well.

Findings

For each run of the experiment, the before and after scores within each of the groups on all of the subtests were compared (t-test, $\alpha = .05$), as well as the scores between the groups using the before score as a covariate (ANOVA, $\alpha = .05$). For the first run through, Group I clearly surpassed the other two groups on all sections of the exam. They increased more dramatically when comparing their prescores to their post-scores, and also when comparing their post-scores to those of the other two groups. The reasons for this seem to lie with the personal bias of the teacher against the methods employed to instruct Groups II and III. Because of this bias, the second run of the experiment was undertaken.

The data for the second run are still being analyzed. At the conference, they will be discussed, compared and contrasted to those of the first run.

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A FRAME OF REFERENCE FOR THEORY, PRACTICE AND RESEARCH IN MATHEMATICS EDUCATION

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In this paper a frame of reference for research and practice in mathematics learning and teaching is described along with general findings from some of the field trials in which its practicality has been appraised.

INTRODUCTION

Drawing from the work of Skemp, Piaget, Dienes, Robert Davis, Lovell, Johannot, Karplus, Alan Bell, and Papert, as well as from personal experience, the writer has been building a frame of reference over the past fifteen years which has proven very successful in guiding the choice, design and implementation of mathematics classroom teaching strategies. It has also helped to generate classroom-based research studies that blend and nurture theory, practice, and dissemination in ways that appeal to teachers. The following is an attempt to communicate something of the nature and applicability of this eclectic frame of reference.

FOUNDATION: PUPILS' EXISTING SCHEMATA

By the mid 1960's the writer had become very impressed by the importance and applicability of the notion of a schema, a basic concept in the work of Piaget and of Skemp (Harrison, 1969, p.94). Then, as now, he understood a schema to be a cognitive structure which has reference to a class of similar action sequences from past experience (Flavell, 1963, pp. 52-53). A schema can be thought of as the structure common to all those acts that an individual considers to be equivalent. For example, if a child's experience has led him to believe that putting three objects with four objects results in a group of seven objects, he possesses a basic schema that will enable him to understand that "3 + 4 = 7."

Any problematic situation requiring behaviour which is already generally represented in the child's mind is handled by being assimilated to the schema. Learning that "3 + 4 = 7" is assimilated to the knowledge that 3 objects and 4 objects make 7 objects. Furthermore, a child with such an operational schema is in a position to understand that three hundred plus four hundred equals seven hundred without ever having to count out that many beads or matchsticks (Skemp, 1958, p. 70). However, when a child encounters a new situation in which none of his existing schemata appear to be appropriate, new relevant experiences are needed from which he can build new or modified schemata to accommodate to the new circumstances. As with life in general, an individual learns to adapt to a mathematical environment through the interplay of assimilation and accommodation. While it is clear that a child needs the relevant schemata to understand and process situations that are obvious analogues or applications of structures that he has built in the past, it is even clearer that the schemata must be well founded and understood if there is to be successful accommodation to new learning situations. Any teaching-learning design which can ensure that each individual will be able to build the prerequisite concepts, before or while tackling new learning tasks, promises to facilitate effective, enjoyable, useful, and transferable learning. In general, conventional mathematics teaching approaches have fallen far short of the mark in the learning qualities listed, as far as most pupils are concerned.

FRAMEWORK: CONCEPTS AND OPERATIONS BUILT ON REAL EXPERIENCE AND REASON

Piaget once stated that an individual's apparent failure to grasp the most basic concepts of elementary mathematics stems not from a lack of any special aptitude but rather from inadequate preparation with its inevitable, concomitant emotional blocking. He went on to say that the frequent failure of formal education can be traced to the fact that it begins with language, illustrations and narrated action rather than real, practical action. Preparation for mathematics education must begin in the home(and continue in school) with the encouragement of concrete manipulations that foster awareness of basic logical, numerical, and measurement relationships. This practical activity should be systematically developed and amplified throughout the primary grades, leading to basic physical and mechanical experiments by the time secondary education begins (Piaget, 1951, pp. 95-98). Unless a pupil is provided with the appropriate kinds of repeated experience necessary to form basic concepts and operations, verbal and blackboard teaching will lead to rote memorization. The concepts required for learning superordinate concepts will not be formed, and the student will never be capable of really understanding mathematics. (Skemp, 1962, pp. 9, 10; 1960, p.50). In the mid 1940's, Louis Johannot (1947) found that the abstract reasoning of the great majority of adolescents was absolutely artificial, merely pseudo-reasoning. Reasoning had been replaced by learned procedures and method types. The role that active, creative, intelligence played was seen to be no more than a secondary one (Johannot, 1947, p. 113; 1976 translation, p. 101). Small scale trials of Johannot's interview tasks in Calgary in recent years have given ample indication that the situation is little different across an ocean and thirty-five years later.

A concise and yet meaning-filled and useful contrast between the form of "reasoning" Johannot found and that which it is assumed that members of the IGPME (at least) would prefer to see fostered has been offered by Richard Skemp. He has dramatically contrasted instrumental understanding, characterized by using "rules without reasons," with relational understanding, "knowing both what to do and why." An instrumental approach to mathematics is a product of the kind of learning which consists of (or stresses) the learning of rules for proceeding from given data to the correct answers to given questions (the next step is determined solely by the immediate conditions). On the other hand, a relational approach to mathematics requires the learner to build a conceptual structure (schema) from which a variety of strategies for solving any given problem can be produced (Skemp, 1978, pp. 9, 14).

It comes as no surprise that, if mathematics teaching neglects the development of mathematical reasoning structures in favour of a steady diet of narrowly focussed "explanations" and "exercises," the development of relational thinking in mathematics is found to be at a very low level. While mathematical strategies cannot be "given" to pupils, they can and must be built by the student through process-enriched experiences generated by problems much larger than the usual textbook exercises (the South Nottinghamshire Project investigations are excellent examples of the larger problems: Bell, Wigley, Rooke, 1978, 1979).

BUILDING: TEACHING TAILORED TO THE LEARNER

One of the goals of the dissertation project which Louis Johannot completed under Piaget's direction was to determine the extent to which secondary school mathematics teaching was adapted to the developed abilities of the pupils for whom it was intended (Johannot, 1947, p.90; 1976 translation, p.78). Johannot documented a significant gap between the demands of the mathematical content and the operational ability levels of most students, which, of course, has been a frequent intuition among mathematics teachers for a long time. A similar gap was found more recently in a Calgary research project (Bye, Harrison, and Brindley, 1980, pp. 12-16; Harrison, Brindley, Bye, 1981, p. 217) involving twelve and thirteen-year-olds learning about fractions and ratios. There was little doubt that the needed latent abilities existed but it appeared that, in the context of fractions and ratios, the development of relational thinking had been very limited under the traditional approaches that the students had previously experienced.

The "student investigation" teaching approach used in the Calgary project was modelled on that of the South Nottinghamshire Project (Bell, Wigley, Rooke, 1978, 1979). This approach was chosen because the writer's frame of reference included the notion that intellectual development is marked by a gradual transformation of overt actions into mental operations (a key Piagetian concept highlighted by Flavell, 1963, p. 368) and that it is important not to pass too quickly from the qualitative (logical) to the quantitative (numerical) when students are learning mathematics (Piaget, 1972, p. 17)

Another context in which the writer has influenced the development of teaching-learning approaches has been in the Highwood Elementary School Math Lab Project, for which he has been acting as an informal consultant on teaching resources and strategies. The Project Teacher (Marilyn Harrison) has been working with the regular teachers over the past six years to develop student-centered, concrete, mathematics learning activity approaches for six to twelve-year-olds. The frame of reference described in this paper has been used extensively in the selection of manipulative materials and in the design of teaching approaches. There is such a wide range of quality and purposefulness in the manipulative materials available to mathematics teachers in North America that it is essential to be guided in making choices by a consistent, workable theory regarding the nature of the child's learning processes in relation to mathematical structure.

APPRAISAL: ACHIEVEMENT, ATTITUDE, REASONING

Having described a frame of reference for mathematics teaching and learning and having referred to two field trials in which it has been applied, a capsule account is now given of some of the findings from the trials. Just before the Highwood Elementary School Math Lab Project began, the Stanford Achievement Test median arithmetic score for the Grade Three students was five to ten percentiles below the median reading score (which was at approximately the fiftieth percentile). After the first three years of development of the Lab, the Grade Three median arithmetic score was then fifteen to twenty percentiles above that of the reading score (which had remained near the fiftieth percentile). Student, parent and teacher attitudes towards the mathematics learning activity approach continue to be exceptionally high in the sixth year of the project (An extension of the project has been funded for a further two years).

In the Calgary Junior High School Mathematics Project it was demonstrated that a concrete, process-enriched teaching approach can result in significantly improved achievement in, and attitude towards, fractions and ratios for twelve to thirteen-year-olds. This occurred while the development, of general mathematical strategies was enhanced and computational facility was maintained (Bye, Harrison, and Brindley, 1980; Harrison, Brindley, Bye, 1981).

It is felt that the frame of reference described (the main ideas of which have been placed in the public domain by numerous scholars and researchers) shows real and practical potential for making classroom learning more like real life learning and, hence, more effective, more lasting, more useful, and more enjoyable.

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Affective Teachers - an exploratory study

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The importance of affective aspects of mathematics learning is widely recognised, and several researchers have studied these aspects with a particular focus on the negative emotions that often accompany mathematical activity. Hoyles (1982) found that one of the factors that seemed to characterise the mathematical experiences of 14 year old pupils was a focus on 'self' rather than on the task at hand, indeed the nature of the mathematical tasks was rarely mentioned. Hoyles (1980) also found that in these critical incidents in mathematics learning recounted by the pupils, the influence of the teacher appeared significant. Other work on teachers' expectations (eg. Brophy & Good, 1969) and on teachers' attributional interpretation of pupils' behaviour (Wiener 1980) combined to suggest that a study of mathematics teachers and their effects on affective learning in mathematics might prove fruitful. As was described last year (Bishop, 1981) this area is one which has not received the attention it really deserves.

The aim of this research then was to explore the relationships between certain teacher characteristics and their classroom practice. The study is naturally exploratory, the samples are accidental ones, and the results are very tentative.

Methods

The characteristics attended to were assessed by means of an attitude questionnaire, a test of ideographic/normative typing (Rheinberg, 1977) and a critical incident questionnaire. The teachers were in two groups: a group of experienced teachers on an in-service mathematics course, and a group of student teachers undergoing initial training. As well as being assessed, the teachers were observed teaching, and in addition the experienced teachers were interviewed using an interactive repertory grid programme (PEGASUS) in order to elicit their personal construct for describing their pupils (Shaw, 1980).

The procedures and some findings will be briefly described but more detailed information will be presented at the conference.

(a) Attitude questionnaire

This was a standard Likert-type questionnaire. Any differences in scores did not in general appear to relate to differences in practice, but the more 'affective' experienced teachers tended to disagree with, or make no comment about:

1. Mathematics is harder work than most subjects.
2. The basic skill which mathematics teaches is accurate calculation.
3. Mathematics is primarily a tool for use in other subject areas.

(b) Test of ideographic/normative typing

The basis of this test, as developed by Rheinberg (1977) is that teachers can assess a pupil's achievement as 'good' or 'bad' according to either the average level of the class (social reference norm NO) or the pupil's prior performance (individual reference norm TO) or to a combination of both. Lorenz (1982) found that differences in reference norm were related to different teacher strategies in the classroom. For this research reported here the original test was modified slightly for use with English mathematics teachers.

For the experienced teachers:

TO scores varied from -2 to 24, mean 7.7
NO scores varied from 0 to 7, mean 4.5

For the student teachers:

TO scores varied from -4 to 25, mean 10.7
NO scores varied from 0 to 7, mean 2.7

(c) Critical incident questionnaire

This questionnaire consisted of a group of described classroom events and teachers were asked how they would deal with them. Differences in response were noticeable - for example, it appeared that the more 'affective' experienced teachers, in two of the incidents, recommended action which focussed the pupil onto the task (eg. introducing new or different mathematical work) while other teachers tended to suggest disciplinary action, sympathy or help.

Follow-up work with experienced teachers

Five experienced teachers, well known to the researcher, were chosen for their classroom expertise although their effectiveness differed to the extent that two of them were recognised as exceptional in the affective aspects (More Affective Teachers, MAT). Their results on the first three instruments were very similar to those already obtained in the larger groups. In particular, for the Rheinberg test, the MAT had lower NO scores (1 or 0) and larger TO - NO differences (≥ 9). In addition, the personal constructs of these five teachers were explored using a repertory grid programme.

The appeal of Kelly's (1955) theory of Personal Constructs is that it begins with the uniqueness of an individual's interpretation of his or her own experience. In this study the purpose of the repertory grid was to investigate the constructs these mathematics teachers have of their pupils. A triadic elicitation technique was used by means of an interactive computer programme which also carried out a cluster analysis of the grid. As well as being very interesting for the teachers themselves, this analysis showed up certain differences between the teachers, for example, that the MAT tended to relate the construct of 'mathematical ability' with that of 'enthusiasm' while the others clustered it with 'seriousness' or 'conforming and unobtrusive behaviour'. The complete repertory grids and their analysis will be presented at the conference.

Follow-up work with student teachers

The second author concentrated on the performance of student teachers in the classroom, and on the relationship between their skills at creating mathematical involvement and the 'predictor' test results.

After making a survey of existing observation schedules it became clear that no one schedule would be satisfactory. However, while attempting to modify some, it was felt that more 'global' observation would be more appropriate at this initial stage. Accordingly the following procedure was adopted - the student teachers' classes were observed and on the basis of the children's behaviour, work-rate, enthusiasm and interest, the MAT students were then compared with the others in terms of their teaching behaviours. The most important characteristcs which seemed to differentiate the MAT were:

Modelling - they presented (and discussed) affective aspects themselves
Empathy - they seemed willing, and able, to comprehend the child's viewpoint and feelings
Encouragement - they offered verbal, and behavioural, support to the children
Flexibility - they could vary their approaches, models, and explanations
Socialising - they encouraged group and social exchanges

These characteristics were then related back to the 'predictor' tests given in the first term. No clear-cut differences were found but sufficient pattern showed where the next stages of the research should proceed. The conclusions will be presented at the conference.

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CO-OPERATIVE LEARNING IN ACQUISITION OF OPERATIONAL CONCEPTS OF PLACE VALUE IN NATURAL NUMBER

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INTRODUCTION

1.01 This paper concerns aspects of practice which has evolved in the first years of schooling in parts of the County of Hampshire, the United Kingdom.

1.02 It concerns children of ages approximately 6.00 years, whose previous experience of numerical activity has been underpinned by considerable experience of 'logic', to be described.

1.03 Thus the emphasis is on overt activity and on the positive aspects of interactions within learning groups of differing sizes.
Great stress is laid on 'game' approaches, where 'game is synonymous with 'work'.

BACKGROUND TO THE STUDY

2.01 It is at this point that the work of different teachers was able, and still is able, to be followed up.

2.02 Whilst most teachers of young children adopt quite sophisticated strategies so that a section of a class only is involved in mathematics at any one time (known in a popular form as 'the integrated day') others who have not yet mastered the necessary skills, or who are temperamentally unsuited to the complex management/leadership demands of the integrated day', tend to operate with a whole class for mathematics. The latter practice continues that which tends to be characteristic of a paper and pencil approach, often linked to the sedentary/passive model of mathematics learning. The latter approach is recognised to produce a fluency on paper. Its medium to long term contribution to healthy attitudes and to understanding of an enabling kind in relation to structure and problem solving skill formation is now generally believed to be of a low order.

2.03 The whole class approach implies for this age range a 'lockstep' characteristic attaching to the sedentary nature of the activity. The rather individualised nature of the paper-and-pencil approach is popularly supposed by its proponents to be transferable to an apparatus resourced approach.

2.04 Where a sufficiently great amount of apparatus is got together, it is possible to see a teacher recently convinced of values intrinsic to 'apparatus' attempting to lead a large group of children. It is unusual, however, for a teacher to be able to 'prevent' children monitoring activity around them (a form of co-operation) when a directly teacher-directed activity is intended by the teacher.

2.05 A comparison of some outcomes of small-group learning, where co-operativity is actively planned for and encouraged, with outcomes from large group learning intended by a teacher to be directed so that multiple 1:1 teacher-learner interactions are believed to be taking place, has been possible.

THE STUDY

3.01 The area chosen for initial study are those activities leading toward establishment of working concepts of place value.

3.02 A brief description of the sequence of experiences which it is intended a child should have would include:

- idea of a single object (unit)
- idea of a collection of objects of number name three, four, five or six
- idea that such a collection can become a new 'single object' or unit
- idea that any new collection of singles (units) may be 'grouped' into 'new' singles or units
- idea that any new unit can have a name comprised of two components: a grouping and an ordinal description

e.g. first order (of) grouping
second order (of) grouping
third order (of) grouping etc.
and no order (of) grouping (to describe original and unused 'units') *Note 1.*

Note 1. An Introduction to the Dienes Mathematics Programme by Peter L.Seaborne 1975 University of London Press Ltd. ISBN 0 340 09217 3

3.03 The process of grouping:

making first groupings leaving singles/units ungrouped
making second groupings leaving some first groupings ungrouped
making third groupings leaving some second groupings ungrouped
and so on

is repeated application of the idea of

STATE OPERATOR STATE applied to an initial system of ungrouped elements.

3.04 The final state of a collection of groupings of various orders can be symbolised in a variety of ways.

3.05 An initial symbolisation of verbal kind e.g.

two singles
one first grouping
one second grouping
base three

eventually requires transformation into one of a 'written' kind if a visual record is to be made.

3.06 However, a diagram providing a collection of ordered regions *(Note 2)* is another form of visual record of considerable value on to which the original 'operated upon' elements might be placed in their 'ordered' form. Alternatively copies of the resulting grouped elements could be placed in appropriate regions.

3.07 Number names symbolised by numerals can be mapped onto the region *(Note 2)*

3.08 Instead of writing being called for, pre-written numberals on card can be employed so that decisions as to what numeral relates to the contents of each region can be attempted by 'group decision' of children where the formation of such groups has been allowed or arranged.

Note 2:

third grouping	second grouping	first grouping	single (unit)

3.09 The decision-taking process can be entered into by

an individual child
an individual child and teacher
a small group of children

3.10 Thus judgements relating to values accruing from decision-taking within a given milieu can be made by an observer.

3.11 However, the quality of an experience which can depend upon its context, affective results, cognitive processes and so on, can have overall effects which relate to the short, medium and long term.

3.12 The topic under consideration is one which has such a profound effect on everyday, curricular, recreative and vocational aspects of survival that the quality of early, consolidatory, extending and operational experiences should have interactive effects which optimise initial understanding and application of the few key ideas involved.

MAJOR QUESTIONS

4.01 One question to be asked relates to the matrix of experiences which at the time of initial learning in place value contributes optimally to short, medium and long term aims and objectives.

4.02 As an opening gambit one might well speculate as to whether the teacher-child-apparatus experience component gains from the child-child-apparatus component.

4.03 A further speculation might be as to how the intensity of experience of crossing and recrossing conceptual boundaries affects the degree of permanence of value of the initial experience.

4.04 Clearly, as time passes the original model of both the state and synthesis of state of a place value model needs to have the robustness needed for re-use, both as a systems maintenance ploy and an initial state upon which to operate on future occasions when sophistication of the concept is attempted.

INFORMATION SOUGHT

5.01 In the present limited study, children who had experienced the small group/large group/individual child/teacher involvement and those children experiencing a pseudo 1:1 relationship with a teacher (whole class taught) were asked a series of nonverbal and verbal questions on a 1:1 basis. The essence of the questions were:

(1) Show me a base amount of material.

(2) Tell me about it

(3) Show me another base amount of material.

(4) Make a second grouping in base four.

(5) Change that (the latter) grouping to base six.

(6) Show me a third grouping in base three

(7) Make 213 base four

(8) Put each number/symbol (on card) next to/below/near the grouping which it matches, in the number 213 base four.

(9) Use the numbers (on cards) to show me 312 base four. Fit the material to the cards so that the material matches the card.

5.02 The means and general ability ranges of children in each of the two 'treatment' groups were sufficiently similar to justify us making rather general comparisons between the performances of the groups.

SOME TENTATIVE CONCLUSIONS

6.01 Evidence of a simple kind untreated statistically which will be presented appears to indicate that a more complex working environment for young children is superior to the apparently simpler environment on the criterion of generating ideas of considerable fundamentality as evidenced by verbal and non-verbal indicators of learning (as opposed to written indicators). The complex environment to which reference is made includes teacher-child apparatus links compounded with child-child apparatus links together with a positive teacher attitude to child-child co-operativity.

The apparently simpler environment in the main was a pseudo 1:1 teacher-child large class relationship in the handling of apparatus where child-child co-operativity received little or no encouragement.

6.02 Of the questions asked in 4.01 - 4.04 a reasonably satisfactory answer to 4.01 is a matter for a mixture of collated results from the increasing number of teachers sharing a communality of philosophy which supports a view that the synthesis of small-based systems of numeration is a particularly effective component of approach. Additionally the work of researchers prepared to set up and continue longitudinal studies of children having varying components of initial and continued experience is likely to lend qualified support to intuitive views expressed here.

6.03 On the basis of the rudimentary evidence collected in relation to 5.01, it would seem that a child-child-apparatus component of experience considerably reinforces teacher-child-apparatus experiences in effective adaptation of behaviour in the desired direction.

6.04 The ideas implied within 4.03 suggest that the linkage or entangling of emotions can have strong effects on at least short to medium term adaptation of behaviour as an aspect of learning. When immediate recall has faded the emotional overlay in relation to the structure forming activity leading to establishment of ideas of place value will tend to have an incremental influence on the 'permanent' value of the initial experience.

6.05 The question of 'robustness of model (or models)' for re-use clearly is at the root of positive attitudes to key ideas. The linkage of cognitive and affective components of behaviour adaptation and the balance of these factors for optimal progress in learning in an area of fundamental importance no doubt bears a relationship to the communality of models available and the possibly increasing value (with learning experience) of idiosyncratic versions of such models.

Heuristic Mathematics Education.

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1. Introduction.

One of the major and interesting didactic questions in mathematics education is the *transfer* of the knowledge and skills the students have acquired. This means the extent to which the students will succeed in utilizing their knowledge and skills in other situations than those of the context they have been taught. That other situations may occur both in mathematics and in one of the many specific domains in which mathematical methods are used or in daily life.

What is the cause of the phenomenon so often recorded, that relatively few students succeed in utilizing the acquired mathematical knowledge and skills in problem situations differing from the types so familiair to them? May this be put down to how those concepts and skills have been taught and learned? Would it be possible to organize Mathematics Education in such a way that students learn to solve problems in an unfamiliar context more satisfactorily? These are essential questions for the optimizing of Mathematics Education and the functioning of the acquired mathematical knowledge and skills in applications.

Heuristic Mathematics Education as is discussed in this article aims at both the study of mathematical theory and the increase of the ability of students in solving problems or in approaching problem situations. The essential feature of those problems or problem situations is they are relatively new to the students. They admit of no solution in the usual way by means of procedures studied before. The students are presented with these problems not only in the application stage of the learning process, but as a rule, these problems are the starting-point for learning new mathematical concepts and procedures. In heuristic mathematics education great attention is paid to the *explicit* teaching of *general methods of thinking* in problem analysis and problem solving. Specific algorithmic solution procedures are only taught if a clearly demonstrable long-term educational goal is pursued.

2. Analyses of problem solving processes.

Research into the developments of solution processes has a long history. From his own experimental data Selz has developed a theory on the basic operations of thinking (Selz 1922, Frijda and De Groot 1981). Duncker studied and reported on solution processes primarily with regard to mathematical problems (Duncker 1935, 1948). In his analysis of the chess player's thinking and "intuition", A.D. de Groot described the problem solving processes of experts (grandmasters and masters) in a theoretical framework based on Selz's theory. (De Groot 1965, Frijda and De Groot 1981). In a more recent investigation many findings of Duncker's on problem analysis have been confirmed (Pushkin, 1972). In the early sixties H. Simon's work in the field of human thinking i.a. was a strong stimulus for the development of the information processing theory and the reorientation of American cognitive psychology. In this research ample use is made of the expert-novice paradigm. The way in which experts in a specific field solve problems is compared with the approach of problems by novices. In the Sovjet Union V.A. Krutetskii and his staff have observed students, aged 8 to 18, for many years in order to find the characteristics of the mathematical abilities of the clever solvers compared with the less talented ones.

The psychological concept *"mental image of a problem situation"* is very useful in this description of Heuristic Mathematics Education Didactics. In short that "mental image" can be defined as the problem the way it presents itself to that specific solver with all the ideas, associations and anticipations belonging to it. At first that mental image will be very vague with annoying associations but in the course of the solution process, it may develop further and enrich itself till the solution is found. At the first examination, at the problem analysis and while being at work on the problem the relevant knowledge for that specific problem has to be activated.

During the solution process the mental image of the problem situation is *developing*. While the solver is engaged on a solution suggestion the problem statement is refined and specialized. At any moment in the thinking process this mental image exists and shows a specific developmental stage, of which the solver is not always conscious. (Frijda and De Groot 1981.) Attempts of gifted students to solve a mathematical problem are mostly consciously organized in accordance with a specific scheme and often appear to occur in the form of a mental experiment.
"If I should....., I.....". (Krutetskii, 1976.)
At the lowest level only, those attempts are guesswork, at which the students do not realize why attempts are made and what the results may be.

3. Organization and quality of required knowledge and skills.

From the analyses of the solution processes it has appeared that the way in which the knowledge of the domain the problem calls upon, has been organized in long-term memory and the coherent quality of the mastery of those concepts, facts, theorems, rules has great influence on the development of the solution process. That quality is closely connected with *the entire network of concepts, principles, rules and procedures* in that specific domain the problem solver has to appeal to. Helping to build a rich network or scheme by means of education deserves great attention. (See e.g. Greeno, 1980; R.E. Mayer 1974, 1977, 1978.)

In practical mathematics education building coherent schemes often clashes with training in specific techniques (algorithms) and splitting the subject matter into little units. Teachers and authors of textbooks are apt to prevent learning difficulties by dividing complex tasks, ideas or subjects into series of little, isolated units, each of which may present fewer difficulties to the students, but strengthens the *fragmentary nature* of the organization of mathematical knowledge in the students memories.

Another aspect of the quality is the *meaning* the concepts and operations to be used may have or may not have for the solver. This meaning is essential to forming an adequate mental image. It is a well-known phenomenon in mathematics education that pupils do formally know and manipulate mathematical concepts and operations, but that these concepts do not mean anything to the students. Pupils thoughtlessly manipulate algebraic symbols, pupils (and students) reproduce abstract definitions without being able to give but one example. "This nonunderstanding performance breaks down as soon as any stress is put on it. If students are given a problem in a slightly different form than they are used to, they won't be able to do it" (R.B. Davis 1980).

4. Characteristics of problem analysis.

It stands to reason that in heuristic mathematics education problem analysis is the explicit subject of teaching. During problem analysis the initial mental image of the problem situation develops and knowledge from memory is mobilized. During problem analysis - a central activity in heuristic thinking - the solver surveys the problem more or less consciously. Research into the data (*situation analysis*) and into what is asked (*goal analysis*) results into the comparison between the given situation and the goal (*conflict analysis*). An initial mental image is formed of what is the point in the problem.

In our research into solution processes of first-year maths students we often

signalize the absence of a systematic problem analysis. Making immediate use of a solution procedure often retards further progress, because that first idea is always returned to. (W.C. Doornbos and A. van Streun 1981).
In his studies of solving technical problems Vaags comes to the conclusion that "weak" problem solvers try out solution schemes successively for very specific problems, in which the application of a more general method of thinking (as a conscious problem analysis) with them is retarded by training in specific techniques. (Vaags, 1975)
Krutetskii (1976) described a similar approach in case of "weak" students.

"Experts" approach a problem in broad outline using rather vague terms and pictures and then follow it up with a process of successive refinements. Expert knowledge appears to be structured in such a way that time and again they are able to utilize the same information in rapid interaction at different levels from overall down to the minutest details. (Larkin and Reif, 1979; De Groot 1965; Krutetskii 1976.)
It seems essential to heuristic mathematics education didactics to teach students to approach problem solving hierarchically both in broad outline and in details. Connected with this students will also be taught to describe their knowledge at difference levels from overall down to the minutest details. Sometimes it appears that vague, verbal or visual descriptions are extremely powerful expedients in making the essential early decision on the solution path.

Present-day mathematics education has trained the students in such a way that they are always intent on acquiring solution procedures as soon as possible for very specific types of problems, which gives little scope to the development of productive ability and strengthen the inability in case of problems difficult to recognize at once. (Biermann, Bussmann, Niedworok, 1977.) Such a "reward directedness" is promoted by doing long series of similar exercises and being rewarded for that performance and the experience that success may be achieved at short notice by using tricks learned by heart. (J. van Dormolen, 1975.) The students who were interviewed during the Madison project, stated that "one does mathematics by following directions and not by being clever, nor by thinking about what one is doing." (Davis and Mc. Knight, 1980) So first of all problem-oriented mathematics education needs interesting, motivating or challenging problems and a favourably affective classroom "climate".

In *situation analysis* the solver tries to make sure he knows the given concepts and relation and examines the direct consequences of the conditions and interrelation of the data.

This examination of the situation is mostly indirect. Finding procedures of approach, discovering properties of problem situations, trying to find relations relevant to the solution is the major point.

Correct solution ideas often develop during *goal analysis* when the solver wonders what exactly has to be demonstrated or calculated. The nature of goal analysis strongly depends on the type of problem.
In problems "to prove" the goal has already been stated and it is quite possible to reason back from the object. Problems "to find" such as finding the solution of an equation or calculating the length of a line segment, require of different kind of activity in goal analysis. And in open problems goal analysis often includes probing the entire field of possible goals corresponding with the given conditons.

Already during situation anclysis the solver is mindful of the goal of the problem and the conditions of the problem in goal analysis.
Conflict analysis is characterized by switching the data from the problem situation to the goal. Closely related to this is the procedure called "means-end analysis" in information-processing theory. (Heller and Greeno, 1979; Simon 1980, 1981.)

5. Heuristics and Algorithms.

Besides the *problem analysis methods* of thinking *heuristics* and *algorithms* are also labelled methods of thinking. *Heuristics* are procedures which may help in problem analysis and in developing the problem, in other words they contribute to forming and developing the mental image of the problem situation.
Algorithms are methods, which, if applied correctly, will certainly lead to the solution.
As for mathematics they mostly belong to the professional knowledge of a specific sub-domain.
In heuristic mathematics education only those algorithms are taught which serve a long-range goal. Algorithms often being useful in the final problem-solving stage. Students will have to know the purpose of such an algorithm and have to be able to discover this as a specialization of a general method of thinking or as a curtailment of a frequently applied solution procedure. Discovering algorithms may be done quite problem-directed.

As has already been said heuristics play a role in problem analysis and problem development. Application of heuristics helps in narrowing down the logically

possible search-domain and in forming a hypothesis concerning the final result (in complex problems an intermediate result). L.L. Gurova (1972), Ju. Kuljutkin (1970,1975) and G. Polya (1946) state that heuristics are methods to reduce the number of solution procedures. Examples of heuristics are:
Examining examples. Drawing a picture, a graph. Converting to a well-known case. Temporarily dropping a specific conditon of the problem. Breaking the problem into subproblems. Examining extreme cases. Examining analogous problem situations. Examining specific cases.

6. Teaching methods of thinking.

In the "Heuristic Mathematics Education Research Project", an educational experiment aiming at teaching mathematics in the 4th grade of a secondary school, which is being prepared by the research team for mathematics didactics of the Mathematical Institute of the State University of Groningen, the following educational strategy has been chosen. Existing subject matter (and textbooks) are being examined in connection with the aspects mentioned earlier and enriched with problems, in which *something general* has to be discovered or problems that may lead to learning *general methods of thinking*. With the aid of these problems the students can learn from their fellow-students which methods of thinking they can use. In doing this it is all-important to have the students state their approaches, procedures, their difficulties and the solution strategies employed by them. By taking a *retrospective view* of their own solution paths and those of others the students will get the opportunity to weight the solution procedures and the results. This *reflecting* on their own approaches, learning, their own mathematical activities, will needs lead to an operational instruction drawn up by themselves and individually determined.

7. The presentation on the Congress.

During the presentation of the VI thh 8 ME-Conference the application of the problem-solving model will be discussed for an example of the protocol-analysis from research to problem-solving processes of first year mathematics students at the State University of Groningen (W.C. Doornbos and A. van Streun, 1981).

H. ASSESSMENT

TESTING MATHEMATICAL PROBLEM-SOLVING

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The purpose of the following three papers is to outline the development of a set of superitems designed to assess mathematical problem-solving, and to report on the content and construct validity of the items. A "superitem" is a set of test questions based on a common situation or stem. In this project, a pool of mathematical situations and a set of items for each situation were designed to provide information about students' qualitatively different levels of reasoning ability. The items for each situation were designed to elicit increasingly complex levels of reasoning from the students. The structure was based on Collis and Biggs' SOLO taxonomy (1979). The items were prepared to be administered to students of 9, 11, 13, and 17 years of age.

In the first paper, Kevin Collis outlines the rationale for the item development part of the project. In addition, the evidence gathered to support the content validity of the items is reported. In the second paper, Murad Jurdak presents the rationale and data to support the hierarchical structure of the items. This evidence strongly supports the construct validity of the items. Then in the final paper, Thomas Romberg argues that from their responses students can be grouped into interpretable groups. This, too, adds to the strong evidence about construct validity.

The project was funded by the Education Commission of the States (with funds supplied by the National Institute of Education). The resulting items are to be used in future National Assessment of Education Progress (NAEP) studies in mathematics.

THE SOLO TAXONOMY AS A BASIS OF ASSESSING LEVELS OF REASONING IN MATHEMATICAL PROBLEM-SOLVING

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The question being addressed in this study was: Can a set of superitems be constructed so that responses will reflect "level of reasoning" used by the subject matter with respect of mathematical situations?

In some tests, items come in groups, e.g., paragraph-reading tests with several questions on each paragraph, or table-reading tests with several items on each table. The problem situations or stems, the paragraphs and tables in the example, contain considerable information. The sets of questions with the stem are called superitems (Cureton, 1965), a term chosen to emphasize that differences among respondents in comprehension of a stem may produce correlated errors of measurement between items for the same stem.

During the 1970s, Wearne and Romberg (1977) developed several versions of a superitem test of mathematical problem-solving for elementary school children. Those tests were designed to produce three scores: a comprehension score, an application score, and a problem-solving score. These tests were then used in several studies carried out under Romberg's direction. In all, these tests proved to be useful in providing information about a student's mastery of the prerequisites for the problem-solving question. However, the problem-solving items in the Wearne-Romberg tests do not yield information about the level of a student's reasoning with regard to each problem situation. For the pool of items constructed in this study, a recently developed taxonomy based on the "structure of the observed learning outcome" (SOLO) (Biggs & Collis, 1982) was used as the blueprint for the development of superitems with four questions.

The SOLO Taxonomy, General

From different starting points, several groups of researchers (Biggs & Collis, 1982; Marton, 1981; Case, 1980; Fischer, 1980) have devised similar models for the development of intellectual functioning in children and young adults. The one by Biggs and Collis provided the basic theoretical underpinning for developing the technique for assessing reasoning in mathematical problem-solving.

Biggs and Collis began by setting themselves the task of giving teachers actual classroom examples of levels of cognitive development which teachers would recognized from their interactions with children in classroom situations. They planned to do this across several curriculum areas such as, mathematics, English, history, geography, economics, and foreign languages, using examples obtained from previous research, from practicing teachers, and from original surveys conducted in schools. Their analysis of the responses from these various sources made it clear that they were in fact dealing with two phenomena. The first was what they chose to call the Hypothetical Cognitive Structure (HCS) and the second the Structure of the Learned Outcomes or Responses (SOLO).

The former was closely related to the existing notion of Piaget's stages of cognitive development [Sensorimotor (birth to 2 years); Intuitive/Preoperational (2-6 years); Concrete Operational (7 to 15 years); Formal Operational (16+ years)] in which each stage has its idiosyncratic mode of functioning and, as far as intellectual development is concerned, its own set of developmental tasks. The latter, on the other hand, was concerned with describing the structure of any given response as a phenomenon in its own right, that is, without the response necessarily representing a particular stage of intellectual development. Examples of the nature of the elements and the kind of manipulations available at each of the stages are as follows:

(a) Sensorimotor: the elements are the objects in the immediate physical environment and the operations involve the management and coordination of motor responses in respect of these objects.

(b) Intuitive/Preoperational: the elements become signifiers (words, images etc.) which stand for objects and events and the operational side involves the manipulation of these in oral communication.

(c) Concrete Operational: the elements develop from mere signifiers to concepts and operations which are manipulated using a logic of classes and equivalences; both elements and manipulations being directly related to the real world.

(d) Formal Operational: the elements are abstract concepts and propositions and the operational aspect is concerned with determining the actual and deduced relationships between them; neither the elements nor the operations need a real world referent.

The structure of the learned responses which occurs within each stage becomes increasingly complex as the cycle develops. Uni-structural responses represent the use of only one relevant aspect of the mode; multi-structural, several disjoint aspects, usually in a sequence; relational, several aspects related into an integrated whole; extended abstract takes the whole process into a new mode of functioning.

To illustrate this structure, suppose a class had been given a lesson on the formation of rain and the teacher asks the question: "Why is the side of a mountain which faces the coast usually wetter than the other side?" The following responses may be obtained from students:

1. "Because it rains more on the coastal side."
2. "Because the sea breezes hit the coastal side first."
3. "Because the sea breezes contain water vapour and they first strike the coastal side and so it rains on them and after that there's no rain to fall on the other side."
4. "Because the prevailing winds are from the sea and they pick up moisture and as they meet the mountain they're forced up and get colder, the moisture condenses, forming rain. By the time the winds cross the mountain they are dry."
5. "This is likely to be true only if the prevailing winds are from the sea. When this is so, the water vapour évaporated from the sea is carried to the mountain slopes, where it rises and cools. Cooling causes the water vapour to condense and deposit. Not only is the wind now dryer, it is then carried up the mountain further, is compressed, now warms, and thus is relatively less saturated than before: the effect is similar to the warm climates experienced on the Eastern slopes of the Rockies in Canada in winter. However, all this makes assumptions about the prevailing wind and temperature conditions; if these were altered, then the energy exchanges would differ, resulting in quite a different outcome."

The five responses listed above represent both an increase in the use of the information available and an increase in the complexity with which it is put together.

Response 1 merely restates the question and fails to engage any relevant information from the previous lesson--thus it would be classified as pre-structural.

Response 2 gives an answer to the question but applies only one of a number of possible pieces of relevant information (or propositions) to the problem--thus it is classified as uni-structural.

Response 3 answers the question and uses several appropriate facts (or propositions) in sequence to justify the conclusion--hence, a multi-structural response.

Response 4 not only answers the question by using several facts (or propositions) but endeavors to integrate them. The introduction of an integrating principle makes this a relational response.

Response 5 shows an entirely new use of the available information (or propositions). Instead of immediately accepting the conclusion implicit in the question the student points out that there are other possibilities and uses the given data to test out some of these hypotheses. This represents an extended abstract level of responding.

The above example applies to the concrete operational mode of functioning. To further clarify the concept let us take another example this time from the sensorimotor mode in relation to the important concept of object permanence, in this case a typical plaything such as a ball.

Pre-structural: If the ball rolls out of sight, the infant does not look for it but acts as though it no longer exists.

Uni-structural: The infant uses only one sense modality, usually sight, to look for the ball that has rolled away; if it is not immediately successful, the infant gives up.

Multi-structural. If the ball is placed under a cushion in full view of the child, he will retrieve the ball by raising the cushion. However, if, again in full view of the child, the ball is removed from under the original cushion and placed under a nearby cushion, the child continues to look under the original cushion. The child uses more than one sense modality but does not coordinate the information.

Relational: The child solves the problem of sequential displacements, first when he sees the displacement later even when the displacement is unseen.

Extended Abstract: In the absence of the ball, the child will ask for it, i.e., "ball?". This response represents a move to a new mode of functioning and the cycle begins again with this response representing the uni-structural stage of the new cycle.

These notions are summarized in Figure 1. (See Figure 1.) Column 1 shows the stage of development or mode of functioning level. Column 2 details the learning cycle as it occurs within each stage and from stage to stage. Column 3 illustrates the extended abstract level of achievement with respect to the development of language at the various developmental stages. Single word sentences being the achievement of the sensorimotor stage; sentences at a mature level of complexity for communication purposes the achievement of the next stage and finally full propositional logic being the end result of the concrete operational stage.

Construction of Mathematical Problem-Solving Items

For this project, we hypothesized that by using the SOLO framework one ought to be able to design items such that a series of questions based on the stem would require more and more sophisticated use of the information from the stem in order to obtain a correct result. This increase in sophistication should parallel the increasing complexity of structure noted in the SOLO categories.

1. Mode (Developmental Stage)	2. Response Structure (Learning Cycle)	3. Use of Language
Sensori-motor (infancy)	Unistructural Multistructural Relational = Prestructural	
Intuitive /Pre-operational (early childhood - pre-school)	Extended Abstract = Unistructural Multistructural Prestructual = Relational	Words
Concrete Operational (childhood to adolescence)	Unistructual = Extended Abstract Multistructural Relational = Prestructural	Sentences
Formal - 1st Order early adult	Extended Abstract = Unistructual Multistructural Prestructural = Relational	Propositions
Formal - 2nd Order and higher order adult	Unistructural = Extended Abstract Multistructural etc.	Propositions of increasingly higher order of abstraction

Figure 1. Response Model of Intellectual Functioning

Thus, as described in the report of the development of superitems (Romberg, Collis, Donovan, Buchanan, & Romberg, 1982), the construction of the items consisted of two parts, writing the stem and constructing questions to reflect the SOLO levels. So that a correct response to each question would be indicative of an ability to respond to the information in the stem at least at the level reflected in the SOLO structure of the particular question, we used the following criteria to write questions:

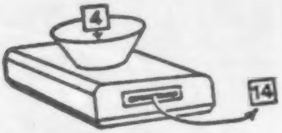
Uni-structural (U)	Use of one obvious piece of information coming directly from the stem.
Multi-structural (M)	Use of two or more discrete closures directly related to separate pieces of information contained in the stem.
Relational (R)	Use of two or more closures directly related to an integrated understanding of the information in the stem.

Extended Abstract (E) Use of an abstract general principle or hypothesis which is derived from or suggested by the information in the stem.

In each superitem, the correct achievement of question 1 would indicate an ability to respond to the problem concerned at at least the uni-structural level. Likewise success on question 2 corresponds to an ability to respond at multi-structural level, and so on.

An example of items constructed in this manner is shown in Figure 2. The stem provides information and each question that follows requires the student to reason at a different level in order to produce a correct response.

This is a machine that changes numbers. It adds the number you put in three times and then adds 2 more. So, if you put in 4, it puts out 14.



U. If 14 is put out, what number was put in?
M. If we put in a 5, what number will the machine put out?
R. If we get out a 41, what number was put in?
E. If x is the number that comes out of the machine, when the number y is put in, write down a formula which will give us the value of y whatever the value of x.

Figure 2. Example of a superitem written to reflect the SOLO taxonomy.

Content Validity

For this item it was intended that, in order to obtain a correct answer, the student would need to process the information in the stem in at least the following ways:

(a) Answer: 4. One piece of information used, one closure required, the information obtainable from either the last sentence in the stem or the diagram—uni-structural response level.

(b) Answer: 17. All the information used in a sequence of discrete closures. The stem is seen as a set of instructions to be followed in order—multi-structural response level.

(c) Answer: 13. All the information used but in addition the student has to extract the "principle" involved in the problem well enough to be able to use it in reverse. The student needs to have an "overview" of the instructions in the stem in order to carry out the appropriate operations--relational response level.

(d) Answer: $y = 3x + 2$. The student has to extract the abstract general principle from the information and write it in its abstract form. This involves dismissing distracting cues, perhaps forming hypotheses and testing them, and zeroing in on the relationships involved--extended abstract level.

Item Writing and Content Validity

To prepare a set of items for use by NAEP, the items were to be administrable to students of 9, 11, 13, and 17 years of age, and they were to reflect five NAEP content categories (Number and Numeration; Variables and Relationships; Size, Shape and Position; Measurement; Statistics and Probability; and Unfamiliar). Content validity was achieved carrying out several content vs. category checks using mathematics teachers, educators, and mathematicians. Some other aspects of validity were taken into account by extensive trialling of the items with samples of the population for which the test was intended; the trials led to several re-writings to take account of readability and comprehensibility of the item stems. For example in the last review, six mathematics educators worked each item, classified each as being primarily in one of the NAEP content categories. In addition, the judges were to identify, for each question in the items, the level of reasoning likely to be employed. Both the items and the questions within each item were randomly ordered for the validity check.

The results indicated a generally high level of agreement for both content and level of reasoning categorizations. The nominal index of agreement (Frick & Semmel, 1978) for the content of judgement was high, $P_o = .68$; and Light's index (1971) for content categories was $\kappa_{p_I} = .70, .53, .83, 62, .67$, and .22 for the content categories with an average of .60. All were statistically significant, only the κ_{p_I} for the "unfamiliar" stems was practically low. For the level of reasoning judgments, all the indices were very high: $P_o = .86$, $\kappa_{p_I} = .85, .76, .79$, and .83 for levels 1 to 4, respectively, with an average of .81. Since these indices were quite high for judgments about content and particularly for judgments on level of reasoning, we decided content validity of the superitems had been demonstrated.

THE CONSTRUCT VALIDITY OF EACH SUPERITEM-GUTTMAN SCALABILITY

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The notion of construct validity implies that the scores on a test can be meaningfully interpreted in terms of related concepts from a psychological theory. The theoretical concepts are called "constructs," and the process of validating such an interpretation is called "construct validation" (Cronbach, 1960). When answers to each question in a superitem are coded (1 = correct, 0 = incorrect), the yield for any student is a score vector of 1's or 0's. Since the structure of the SOLO taxonomy assumes a latent hierarchical and cumulative cognitive dimension, and the response structure associated with any level of reasoning determines the response structure associated with all lower levels, in the sense that the presence of one response structure implies the presence of all lower response structures. The five expected response patterns for each of the SOLO superitems are (0000, 1000, 1100, 1110, and 1111) for the four questions in hierarchical order. These five response patterns are called the Guttman true types (Guttman, 1941). Any deviation from a true type is classified as an error. A measure of the extent to which the observed response patterns belong to Guttman true types was used as one indication of the construct validity of each item.

Test Administration

To gather appropriate information on these superitems, separate group-administered tests were prepared for the 17-year-olds and for the 9-, 11-, and 13-year-olds. Separate tests were necessary because three of the most difficult items were judged appropriate only for 17-year-olds and the tests for the 17-year-olds included the questions for all four levels of reasoning: uni-structural (U), multi-structural (M), relational (R), and Extended Abstract (E), whereas the tests for the three lower age levels did not include the extended abstract question.

Five test forms of seven superitems each were created for each of the two age groups (17-year-olds and 9-, 11-, and 13-year-olds). A central Wisconsin school district serving a community of 32,000 and the surrounding rural area agreed to provide a sample of approximately 300 students in each age group for the administration of the batteries. The tests were administered during the week of September 14-18, 1981. Test booklets were randomly distributed to students. These data along with follow-up interview data from a small sample were used to answer six questions addressed in the study (see Romberg, Jurdak, Collis, & Buchanan, 1982, for details).

Results

The question being addressed in this paper was: For each item is the pattern of responses a Guttman true-type response? Three indices were used to examine whether the responses of students at each age level for each superitem were true Guttman types. First, a coefficient of reproducibility was calculated:

$$\text{coefficient of reproducibility } (r) = 1 - \frac{\text{total no. of errors}}{\text{total no. of responses}}$$

Any response pattern which is not a true Guttman type is considered an error. Thus, if there are no patterns which are considered errors, the coefficient of reproducibility is 1 and the scale is a perfect Guttman scale. If all response patterns are errors, then the coefficient is obviously zero.

In addition, Proctor (1970) formuated a probablistic representation of the observed data in order to base the acceptability of Guttman method on statistical criteria of goodness of fit rather than judgment and experience. Based on maximum likelihood procedures, a misclassification parameter (p) is calculated. This is based on the predicted distribution of types to the predicted one is investigated by chi-square techniques. A superitem was considered to be a Guttman true type if $r > .85$, $p < .15$, and χ^2 was non-significant at .05 level.

For 17-year-olds, of the 35 superitems, 31 had a coefficient of reproducibility (r) greater than .85. This means that the errors, i.e., deviations from Guttman true types, were less than 15% of all responses for these superitems. The probabilities of misclassification (p) were not more than 15% for 30 of the 35 superitems. Consequently, there was no more than 15% response error for the items, based on the observed frequencies of nonscale types for 30 out of the 35 superitems. The goodness of fit between the actual distribution of frequencies of types and the predicted distribution based on the required probability of misclassification is given by χ^2. The χ^2 was significant at .05 with df = 10 for only six superitems, five of which had a high probability of misclassification. Overall only 4 superitems had practical problems which indicate they do not reasonably reflect the SOLO taxonomy, 2 superitems were questionable, and 29 were satisfactory.

The results of the scalogram analysis for 13-year-olds indicated that of the 35 superitems, 31 had a coefficient of reproducibility (r) greater than .85. The probabilities of misclassification (p) were not more than 15% for 32 of the 35 superitems and no significant departure from the predicted pattern was found for 28 superitems. Overall there were 8 superitems for which at least one negative indicator was found. In summary, there were 27 satisfactory superitems, 3 that are questionable, and 5 that do not reflect the SOLO taxonomy levels.

For the 11-year-olds, for the 35 superitems, 30 have a coefficient of reproducibility (r) greater than .85, 29 do not have probabilities of misclassification greater than .15 and a significant χ^2 was not found for 25 superitems. Overall there were 12 superitems for which at least one negative indicator was found. In summary for the 11-year-olds, there are 26 satisfactory superitems, 4 questional superitems, and 5 which do not reflect the SOLO levels.

The results for the 9-year-olds were similar. For 30 of the 35 superitems, r is greater than .85 and p is less than .15, and for 25 superitems, χ^2 was not significant. In all only 10 items have negative indicators.

Finally, for the 9-year-olds, 27 items were considered satisfactory, 3 questionable, and 5 unsatisfactory.

In summary, for the 32 items that were administered to all four age groups, 20 were satisfactory for all ages. Furthermore, when one examines the questionable and unsatisfactory items across all ages, each appears to have a content validity problem. Only two items were questionable or unsatisfactory for just one age group. For both of those, the problem for 17-year-olds was only with the E question. Thus, when one adds to the base satisfactory items the three superitems only administered to 13-, 11-, and 9-year-olds, we get 25 satisfactory items for those age groups. For 17-year-olds, 29 superitems were satisfactory. In general, this is strong evidence that the superitem format in which terms were constructed to fit the SOLO taxonomy forms a Guttman scale.

Other Supporting Evidence

By contrasting the p values for each level across age levels, a consistent picture of growth can also be shown. The means for U, M, R, and E scales for the 17-year-olds and the UM and R scale for the 13-, 11-, and 9-year-olds for each form are shown in Tables 1 to 4. At each age the decrease in mean performance from U to R or E is consistent. Furthermore, since 13-, 11-, and 9-year-olds took the same forms, a cross-sectional comparison indicates age levels was clear (see Figure 1). Similarly, if one were to look at individual items, the same pattern of differences was apparent. For example, in Table 5, the p values for U, M, R, and E were shown for superitems C3 and D2 for each age group. Although there are differences between both forms and items, the profiles of change across age levels are consistent with the

Table 1

Scale Means for 17-Year-Olds on U, M, R, and E for Each Form

Form	SOLO Response Level				
	n	U	M	R	E
1	61	5.13	5.52	3.02	.64
2	58	6.28	4.45	3.01	1.38
3	63	5.91	5.43	2.29	.80
4	57	6.61	5.88	3.39	.93
5	62	6.48	5.79	3.89	.92

Table 2

Scale Means for 13-Year-Olds on U, M, and R for Each Form

Form	SOLO Response Level			
	n	U	M	R
1	109	6.02	4.41	1.06
2	97	5.20	4.26	2.66
3	95	5.52	4.82	1.71
4	95	5.96	4.06	1.79
5	94	5.31	4.37	1.23

Table 3

Scale Means for 11-Year-Olds on U, M, and R for Each Form

	SOLO Response Level			
Form	*n*	U	M	R
1	82	5.36	3.79	.41
2	72	4.32	2.78	2.06
3	71	4.65	3.65	1.06
4	76	5.22	3.01	.81
5	71	4.36	3.06	.41

Table 4

Scale Means for 9-Year-Olds on U, M, and R for Each Form

	SOLO Response Level			
Form	*n*	U	M	R
1	69	3.86	2.10	.12
2	60	2.95	1.70	1.45
3	60	2.92	1.50	.35
4	61	3.28	1.26	.26
5	58	3.37	2.19	.24

notion that there are latent cognitive levels which underlie the SOLO taxonomy and that performance is cumulative and hierarchical.

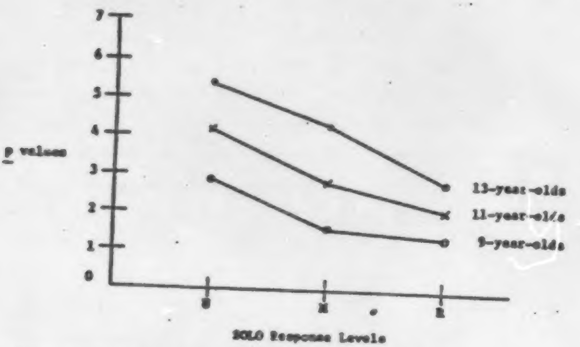

Figure 1. Profiles of p values on the U, M, and R scales for 13-, 11-, and 9-year-olds on Booklet 1 Form 2.

Table 3

p values for Superitems C3 and D2

by Level of Question for Each Age Group

Superitem	Age Level	SOLO Response Level			
		U	M	R	E
C3	17	93.0	54.4	40.4	3.5
	13	88.4	30.5	14.7	—
	11	70.3	33.8	5.4	—
	9	45.9	9.8	1.6	—
D2	17	86.2	74.1	25.9	13.8
	13	89.0	74.3	14.7	—
	11	73.2	53.7	3.7	—
	9	62.3	46.4	0.0	—

THE CONSTRUCT VALIDITY OF AGGREGATED SUPERITEMS--CLUSTER INTERPRETABILITY

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The aggregated scores of students on superitems corresponding to the four levels of reasoning in the SOLO taxonomy provide a basis for a possible natural arrangement of subjects into homogeneous groups. If a student's responses to a set of superitems are all Guttman true-type responses, and if the student is at a particular base stage of development, one would expect the average response pattern across several superitems to reflect that base stage of development. Furthermore, for a large number of students at any age level, one would expect that groups of students with similar response patterns for a set of items could be identified. It is plausible that the profiles of response patterns for the groups can be interpreted in terms of the SOLO taxonomy.

Group Profiles

The profiles which would be interpretable are based on the notions of equilibrations which involve "formation instability combined with a progressive movement toward stablility" (Langer, 1969, p. 93). Cognitive development is seen as "spiral" and, in particular, it is assumed that "to go forward it is necessary to go backward: the first step toward progress is regress" (Langer, 1969, p. 95). From a consideration of this notion, four suggested response profiles for students based on the SOLO superitems for two neighboring levels of performance are shown in Figure 1. The first (X) and the last (Y) steps show stability of performance at neighboring cognitive levels. The steps in between show the regression from X to Y (X→) and the progression from X to Y (→Y). These are seen as steps in the transition between cognitive levels.

In actuality the profiles shown in Figure 1 are ideal. The actual profiles found in this study were likely to be different for two reasons. First, because the questions at succeeding levels are more complex, there would be an increase in the probability of making errors at higher levels. Thus, the p values for Y will be lower than that for X. Second, since the superitems involve different content areas and require students to read the items, either unfamiliarity with the specific content of an item or inability to read the words would depress the patterns shown in Figure 1. If particular P values on X and Y are similar but moderate, students would be reasoning at the Y level on those problems they understood. We decided to indicate this pattern by adding the symbol "+" to its descriptors. In summary, if a student profile for the set of superitems could be grouped with other student profiles and if the groups' average

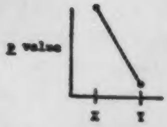
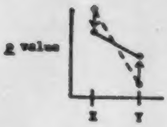
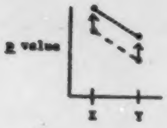
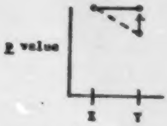
Graph	X,Y *p* values	Slope of line joining *p* value of X with *p* value of Y
 Step 1: Level X (X)	high X low Y	m_1: negative, steep
 Step 2: Regression to Y (X→)	decrease X increase Y	m_2: negative, shallow $\|m_2\| < \|m_1\|$
 Step 3: Progression to Y (→Y)	increase X increase Y	m_3: negative, shallow $\|m_3\| = \|m_2\|$
 Step 4: Level Y (Y)	stable X increase Y	m_4: level $\|m_4\| < \|m_3\|$

Figure 1. Four profiles of *p* values for transition to neighboring cognitive levels.

profile could be judged as similar to one of the four profiles (including depressed profiles) shown in Figure 1, then interpretability in terms of a developmental base was to be claimed.

Procedure

The maximum hierarchical clustering method (Johnson, 1970) was used to partition the students on each form and across forms into homogeneous groups based on score vectors whose four components were the aggregated scores on the four taxonomic levels of reasoning: uni-structural (U), multi-structural (M), relational (R), and extended abstract (E). Before this analysis was carried out, items which failed to reflect a Guttman scale, were omitted. Different possible numbers of cluster groups were considered and then profiles of means for each cluster group on each level of question were contrasted. These profiles and contrasts were then examined to see if they were interpretable. Clusters were first found for each form and then a sample across forms for each age group.

Results

For each of the five forms of the test given to 17-year-olds, the number of groups found varied from 5 to 7 depending upon the form. Twenty-seven of 28 groups over the five forms were considered interpretable. In general, the majority of students at this age are in transition between the M and R levels. A few of the students who took each form have R or higher patterns and a very few have U to M patterns.

Because of this interpretability of clusters across forms, a random sample of 150 students was drawn from the total population. The cluster analysis of the percent correct and information for this sample is shown in Table 1. The seven groups were all interpretable. Fifty-four percent of this sample are from M to R in cognitive level, 31% are above R and 16% are below M.

For 13-year-olds the derived cluster groups for each form varies from 5 to 6. Twenty-three of the 26 groups were interpretable. In general, the group profiles are around M with a few above M and approaching R, and a very few around U. Again, since the groups by form were interpretable, a random sample of 151 students was drawn. The cluster analysis of the profiles for this sample is shown in Table 2. The largest group M comprises 50% of the sample with another 11% being M+. Fifteen percent are at level U; another 15% are above M. Finally, 6% are between U and M and 4% are below U.

For the 11-year-old population, derived cluster groups for each varies from 5 to 7. Twenty-eight of the 31 profiles across forms were considered interpretable. However, as one would expect because of the lower grade level, the number of

Table 1

Percent Correct on Each Level for Cluster Groups
17-Year-Olds, Sample from All Forms

Group	n	SOLO Response Level U	M	R	E	Label
1	25	99	94	79	35	<E
2	22	95	88	32	18	R+
3	41	97	92	58	07	R
4	30	90	76	33	01	M+
5	12	78	73	06	09	M
6	20	86	46	20	01	<M
7	6	45	20	00	00	<U

Table 2

Percent Correct on Each Level for Cluster Groups,
13-Year-Olds, Sample from All Forms

Group	n	SOLO Response Level U	M	R	Label
1	5	.95	.94	.76	R
2	9	.92	1.00	.43	<R
3	8	.91	.67	.50	M+
4	75	.92	.77	.15	M
5	16	.50	.48	.06	M-
6	9	.72	.51	.13	U+
7	23	.94	.44	.02	U
8	6	.63	.14	.00	<U

"depressed" profiles for groups increased. In general, the profiles reflect students in tansition from U to M. Again, a random sample of students (154) was drawn from the population. The cluster analsyis of the profiles for this sample is shown in Table 3. All seven groups are interpretable. Fifty-eight percent of the population reflect a transition from U to M (U+ and +M). Another 29% have reached level M (or M+), 11% below U, and only 9% above M.

Finally, the same procedures were followed for the 9-year-old population. The number of derived cluster groups varied from 5 to 6 depending upon form. In general, the patterns were more difficult to interpret because of the low percent correct for all R questions and most M questions and problems with "depressed" profiles. However, 23 of the 26 group profiles were considered interpretable. For students of this page, the profiles reflect patterns across the U level. A random sample of students of 125 students was drawn from the population.

The clusters formed from the profiles for this sample are shown in Table 4. All six groups are interpretable. Fifty-four percent of the students reflect a pattern around U(+U, U or U+). Twenty-eight percent are at the P level, 18% are nearing the M level.

The consistency and interpretability of the cluster profiles across the forms indicates among other things, the stable influence of cognitive levels of development in the formation of the clusters. The clusters thus formed provide support to the sequence of SOLO levels of responses.

Furthermore, the clusters strongly support the utility of the SOLO response categories over the developmental base stages in Piagetian terms. According to the taxonomy, 17-year-old students should be at formal operational level but most do not operate at the extended abstract level. For example, cluster group (Table 1) has the highest performance on E questions, was judged to be at the E stage of response, and contains only 16% of the students at this age. The majority of 17-year-olds operate around the relational level as seen from the size of clusters 1, 2, and 3 on the relational scale. This suggests that answering questions at the extended abstract level involves more than level of cognitive development.

Similar observations are obvious at each age level. No student profiles are above the hypothesized corresponding level of cognitive development. In fact, most profiles are below the base level of development for an age group. Again, this is strong support for the SOLO levels and their utility in describing responses of students.

Table 3

Percent Correct on Each Level for Cluster Groups, 11-Year-Olds, Sample from All Forms

Group	N	SOLO Response Level			Label
		U	M	R	
1	10	.94	.91	.27	<R
2	4	.73	.46	.40	M+
3	21	.99	.68	.13	M
4	13	.42	.45	.06	M+
5	40	.75	.62	.09	<M
6	49	.81	.36	.02	U+
7	17	.49	.18	.01	U

Table 4

Percent Correct on Each Level for Cluster Groups, 9-Year-Olds, Sample from All Forms

Group	N	SOLO Response Level			Label
		U	M	R	
1	9	.52	.52	.07	M+
2	14	.78	.57	.02	<M
3	18	.80	.36	.06	U+
4	7	.71	.00	.06	U
5	42	.47	.13	.03	<U
6	35	.16	.07	.01	P

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ASSIGNMENT TEST: AN INSTRUMENT FOR QUALITATIVE EVALUATION OF STUDENT MATHEMATICAL ACTIVITY

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A major problem in the study of student mathematical activity in open problem solving situations, is the provision of a suitable theoretical framework against which this activity can be objectively evaluated.

One approach to this problem involves the "atomization" of student activity into small mathematical steps. In this respect the work on artificial intelligence and human simulation helped to clarify some aspects of problem solving and provided models useful for further research. The work of Newell and Simon (1972) describes the major advances in this area.

By comparison, an alternative approach can be described as "molecularization"; that is, the breaking down of the activity no further than what may be reasonably assumed to be performed by the student in one strategic step. In the investigation of student intellectual activity in complex situations the atomic level is less relevant than the molecular one. In particular, the latter is more applicable in classroom teaching evaluation. In recent years researchers have attempted to develop models and measures of student performance in this direction. Bell (1981) reports on evaluating process aspects of the maths curriculum. Schoenfeld (1982) is interested in explicit heuristic training and presents measures for student activity. He uses a paper-and-pencil test. For each problem a list of all plausible approaches is prepared. Then, for every student paper and each approach, information is gathered about "evidence" of the approach, "pursuit" of the approach and "progress" (little, some, almost, solved) toward a solution.

For some years now we have been working in this area; in particular we have developed Assignment Projects which were designed to enhance student mathematical activity (Bruckheimer and Hershkowitz (1977), Zehavi (1979)). An operational framework related to the goals of the Assignment Projects, which allows practical qualitative evaluation of student activity has been developed. This framework has been interpreted in terms of the Skemp model (1979a, 1979b) in a way which provides a deeper investigation of the thought process that yield the solutions to be found in the students' papers. In this paper we shall describe this framework and its relation to the Skemp model.

An Assignment Test in accordance with the goals of the projects and within the framework have been constructed. This instrument can be applied for pre-post and experiment-control comparative evaluation.

Goals of the Assignment Projects and the related framework

The Assignment Projects were developed as an attempt to prepare material which fosters "real mathematical activity". Important aspects of mathematical activity can be identified as investigation, problem solving, proof, generalization and extension. The whole is greater than its parts and, by composition of these aspects we mean "real mathematical activity".

For the development and evaluation of such complex material we definitely need an operational framework. We shall present a framework which combines cognitive and cogno-affective goals. The relation of the goals to these projects has been discussed extensively elsewhere (Zehavi, Eylon, Ben-Zvi and Bruckheimer, 1982).

Formulation of the goals

After working a number of Assignment Projects, we would expect improvement of student responses in the following aspects.

N - non-avoidance of unfamiliar problem.

Faced with a problem which appears unfamiliar, students will not avoid it, but will make a serious attempt, either by a spontaneous activation caused by the problem context or as a result of reflective thought.

S - quality of solution
- SA - quality of answer
- SP - quality of procedure

SA - Instead of being satisfied with the first answer(s) obtained, there will be a conscious attempt toward a complete solution, e.g. explicit consideration of (all) possible cases, and the establishment of a pattern (if possible).

SP - Rather than the use of particular examples only, there will be evidence of the use of higher order procedures in building and testing the answer; e.g. use of general proofs and complete arguments.

ES - extension of solution
- EA - extension of answer
- EP - extension of procedure

The response to a question will show evidence of an attempt to go beyond the immediate and routine and will extend the answer or the procedure.

EQ - extension of question

Student will pose and (try to) answer problems suggested by a given question. Their problems will be "interesting" and "original", rather than just "changing the numbers".

These goals imply a method for evaluating higher order student mathematical activity. The first thing, which is simple, is to find out the degree of non-avoidance (N) of unfamiliar tasks.

To judge the quality of student solutions (S), each response is classified on a 3-point quality-of-answer scale (SA):

1. primitive 2. partial 3. full

and a 3-point quality-of-procedure scale (SP):

1. trial and error 2. incomplete argument 3. complete argument

The quality of solution is considered as a combination of answer and procedure. For each of the two scales, we shall need a more refined classification, which we use in the second stage of the analysis.

The extension goals (ES/EQ), by their very nature, occur only in a minority of the responses. We have some preliminary examples of responses in this category, but we expect to do more work in this area.

Interpretation in terms of the Skemp model

The evaluation method described above implies a substantive solution (SA/SP) analysis. If the problem is sufficiently "rich" all nine combinations of SA/SP may occur.

Based on the Skemp model (1979a, 1979b), we can offer a deeper interpretation of this analysis. Skemp distinguishes between different kinds of understanding, *instrumental, relational* and *logical*. We shall try to relate the quality of solution to the different kinds of understanding.

Table 1. The answer/procedure analysis in terms of different kinds of understanding

Answer SA / Procedure SP	1 primitive	2 partial	3 full
1 trial and error	I instrumental	IR instrumental-relational	R relational
2 incomplete argument	IR instrumental-relational	R relational	RL relational-logical
3. complete argument	R relational	RL relational-logical	L logical

As illustration of the classification in the table and the analysis of student responses, consider the following item: *Which numbers, when substituted in $3x + 1$ will give even numbers?*

If the solution consists of just odd numbers found by a "guess and check" procedure, the solution is classified as I. If a pattern ("all odd number") is established, then the solution is classified as IR. The addition of a proof of a sufficient condition, changes the classification to R. A solution is classified as L if it gives the full answer, i.e. numbers of the form $\frac{2n-1}{3}$ (n-integer), combined with a proof that this is both sufficient and necessary.

Skemp's major hypothesis is that intelligent learning can be explained in terms of a second-order goal directed system Δ_2, acting on a teachable first-order Δ_1.

We do not have direct access to Δ_2, but infer its activity from our observation of Δ_1. It is reasonable to assume that Δ_2 may be activated to obtain movement along the chain I - IR - R - RL - L. In the process of movement along the chain, steps may or may not be skipped. An implication, however unlikely, of this suggestion is that a student may get a fully logical solution in a Δ_1 mode. Δ_2 may also be activated when a student is "stuck"; that is, he is caught in a particular cell (Table 1) and never gets out. One student, for example, obtained a numerical answer by trial and error and then commented that she had tried hard but failed to "explain the answer in terms of the formula". No doubt, she was in a Δ_2 mode, but instrumental.

Assignment Test

The framework we have described can be used to analyse student work on the Assignment Projects, but by itself does not distinguish between the mathematical maturity achieved as a result of regular curriculum studies, and the special contribution of working on the projects. For this we need a relevant instrument - an Assignment Test, whose design and application we shall describe in the remainder of this paper. The goal-framework as described for the treatment was also used as a basis for preparing and analysing the Assignment Test.
The design of the test is governed by the frame work, shown in Table 2.

Table 2. The design of the Assignment Test in terms of the goals

goal \ assignment		1	2	3	4	5
N-nonavoidance				x	●	●
S-solution	SA-answer	x	●	●	●	x
	SP-procedure		●	●	●	
ES- extension of solution		x	x	●	x	x
EQ- extension of question		x	x	x	x	●

● means that the relevant goal is emphasized in the assignment.

The particular pretest we used, contained the following assignments.

Assignment No. 1 : *Given that $f(x) = ax^2 + 3$ and $f(3) = 30$, find $f(10)$.*

Assignment No. 2 : *Which numbers when substituted in $3x + 1$ will give even numbers?*

Assignment No. 3 : *Given the expressions $x^2 - y^2$, $(x - y)^2$, find pairs (x, y) which when substituted in the above expressions give the same result.*

Assignment No. 4 : *Given the equation $b(2x + b) = 175$, where x and b represent positive integers, what can you say about x and b?*

Assignment No. 5 : *Pose and try to solve question(s) similar to those which appear above.*

Discussion

The test was given to 498 ninth grade students. We shall discuss in detail the analysis of the responses to assignment No. 3 in terms of the framework. The students have not met the question in this form before, but it is strictly related to curriculum. A mere 27 students avoided it altogether.

The full answer is $\{(x, y) / x = y \text{ or } y = 0\}$

203 students gave the generalization $x = y$ only, and no more than 20 students gave $y = 0$ only, 24 gave the full answer.

Answers were obtained either by an inductive approach, investigating the two expressions, or by deductive approach considering the equation $(x - y)^2 = x^2 - y^2$.

Table 3. Answer/Procedure distribution (in %) for assignment No. 3

Answer SA / Procedure SP	1 primitive	2 partial pattern	3 full pattern	
1 trial and error	21.4	27.7	2.2	51.3
2 incomplete argument	11.9	9.5	0.6	22.0
3 complete argument		19.6	1.8	21.4
	33.3	56.8	4.6	94.7

To illustrate the analysis which gave us Table 3, we shall bring one example for each cell. Within any one cell - there is of course, a variety of responses.

SA = 1 SP = 1

Numerical examples for which $x = y$ only, or $y = 0$ only.

The numbers were checked in the expressions.

SA = 2 SP = 1

Generalization $x = y$, justified the checking numerical examples.

SA = 3 SP = 1

Generalization $x = y$ or $y = 0$ justified by numerical examples

SA = 1 SP = 2

$(x - y)^2 = x^2 - 2xy - y^2$ (error) $\mid x^2 - y^2$

the two expressions differ in $-2xy$; in order to get the same results $-2xy$ needs to be zero. Take the pair (0, 0) for example.

SA = 2 SP = 2

$$(x - y)^2 = x^2 - y^2$$
$$x^2 - 2xy - y^2 = x^2 - y^2 \quad \text{(error)},$$
$$-2xy = 0$$

x is zero and y any number or y is zero and x any number.

SA = 3 SP = 2

Both $x = y$ and $y = 0$, with general proof for only one of them.

SA = 1 SP = 3 Did not occur

SA = 2 SP = 3

$$\begin{tikzcd} & x = y \arrow[dl, Leftarrow] \arrow[dr, Rightarrow] & \\ x - y = 0 \arrow[d, Leftrightarrow] & & x^2 = y^2 \arrow[d, Leftrightarrow] \\ (x - y)^2 = 0 & & x^2 - y^2 = 0 \end{tikzcd}$$

SA = 3 SP = 3

$$x^2 - y^2 = (x - y)^2$$
$$x^2 - y^2 = x^2 - 2xy + y^2$$
$$2xy = 2y^2$$
$$xy = y^2$$
$$y = 0 \text{ or } x = y$$

Conclusion

Mathematical activity may possibly be described as a continuing process of interchange between the activation of Δ_1 and Δ_2, to achieve the desired goals. Cogno-affective obstacles may impede activity in Δ_1, and cognitive difficulties may impede the activation of Δ_2.

The Assignment Projects, which provide rich problem situations directly related to the junior high school curriculum, were designed to enhance achievement at the higher cognitive levels. In order to measure if this enhancement takes place, we have designed and validated an Assignment Test which faithfully reflects the goals and methods of the Assignment Projects. Progress on the SA/SP scales would be reflected in tables like Table 3, by a significant "population shift" in a direction between "south and east". This will be investigated in the next stage of the research project, of which this paper is one further stage.

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I. ERRORS

Use of knowledge and error detection when solving statistical problems

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It is an important criteria of successful problem solving in educational contexts that the solver has as few errors as possible left in his or her final solution. To fulfill this criteria the solver should either not make any errors or detect the errors made. The present study focuses on solvers' detection of their own errors in statistical problem solving but it also investigates some factors relevant for the occurrance of errors when solvers choose solution methods for the main substeps of the problem.

When solving a problem the solver has available at least two kinds of knowledge resources. The first of these is knowledge about solution methods and includes the name, application conditions and steps of different solution methods in the problem domain. The second consists of knowledge about the meaning of concepts and the different components of solution methods in the problem domain.

Problem solvers may differ as to how they utilize their knowledge resources. For example, when choosing a solution method the solver may let the choice be determined by a careful, planned deliberation of the pros and cons of different solution methods. Alternatively, the solver may let the choice be determined by whatever solution methods spontaneous retrieval processes bring into his or her short-term memory. The present paper evaluates the frequency and success of these methods when used by novice solvers in statistics.

Turning next to solvers' detection of errors we first note that, generally speaking, error detection can be viewed as consisting of two parts, i.e.

(1) triggering of the error detection mechanisms, i.e. the initiation of an evalution process

(2) later steps taken in the evaluation process. These may eventually include detection and elimination of an error.

In analogy with the processes of selecting solution methods it can be noted that there are two basic ways that error detection processes may be triggered. Firstly, they may be invoked centrally irrespective of the actual performance produced. Such processes have been suggested by Hayes and Flower (1980) and Allwood and Montgomery (1982). Secondly, error detection processes may be invoked by spontaneous triggering due to some feature of the performed activity or produced result.

It is of interest to consider more specifically why the spontaneously triggered error detection processes are invoked. Hayes and Flower (1980) have suggested that error detection is initiated when a match occurs in short-term memory between representations of specific errors and the same error made in the emitted performance. The idea of one or many monitors which detect certain classes of errors suggested by Norman (1981) also seems to belong to this category.

In this study it is argued that error detection processes can also be triggered because of some perceived discrepancy between the produced activity or result and a held and appropriately activated expectation. This notion can be seen as a generalization of Carpenter and Daneman's (1981) result that in reading, error detection processes are triggered because of a perceived inconsistency between an earlier interpretation of the text and the presently interpreted text.

Method

16 subjects (13 male, 3 female) studying first year statistics at the University of Göteborg, Sweden were each asked to solve two statistical problems, as described below. All experimental sessions were conducted on an individual basis. Each subject was requested to vocalize all thoughts including those that seemed unimportant while attempting to solve the problems.

The first of the two problems given to the subjects was a time-series problem. Given some sales values for each quarter during three years, the task was to analyse the development of sales by choosing an appropriate model and calculating the seasonal index and the sales values adjusted for seasonal influence. The second problem was a regression analysis problem which asked the solver to calculate a regression line of the size of a harvest on the amount of fertilizer used and to calculate the linear correlation between the two variables.

The problem also asked for the mean increase in harvest for each unit of fertilizer used as well as the expected size of harvest on an area if a specified amount of fertilizer was used. (The problem instructions given to the subjects and the solutions to the problems are given in the original paper which can be obtained from the author.)

Results and discussion

Subjects' think-aloud protocols were analyzed with the help of various coding systems. As the interjudge reliabilities vary from .83 to .92, the median being .88, they can be regarded as acceptable.

All in all, the subjects made a total of 327 errors. Twenty-nine of these involved not finishing a main substep in the problem. Sixty-eight errors were made when selecting a solution method for one of the main substeps in the problems. Fifteen were higher level mathematical errors, such as errors made when performing algebraic manipulations of equations. The majority of the errors, 201, were execution errors, i.e. low level errors such as computation errors, errors where a part of the calculation was done erroneously after it had already been performed correctly once or copying errors.

An analysis which investigated the relation between the occurrence of errors and subjects' use of arguments when selecting a solution method showed that subjects did not put forward any argument at all for their choice of solution method on two-thirds of all selection occasions. It was also found that erroneous choices were more common when subjects did not put forward any argument to support their choice compared to when they gave factual arguments. Most choice situations were found to involve only one solution alternative but the number of alternatives considered did not directly affect the chances of making an erroneous choice. However, it was also found that subjects more often gave factual arguments for their choice when the choice situation involved two or more alternatives as compared to only one.

Turning next to subjects' detection of their own errors we first note that by definition, all error detections occurred in episodes where subjects evaluated their solution (reflective episodes), although all reflective episodes did not involve detection of errors. Subjects' reflective episodes were classified into four types. Of these, Direct Detection episodes (an error is detected in one step) and Error Suspicion episodes (subject perceives negative symptom in solution) were the most common and were found to be more common than Routine

Check (centrally invoked check) and Affirmative Evaluation episodes. Most of the subjects' 327 errors (78%) were found to have contributed to some solution part which triggered a reflective episode. About two-thirds of the undetected errors were not connected to a reflective episode. These results can be compared to the finding by Montgomery and Allwood (1978) that subjects rated 52% of their erroneously solved substeps as completely correctly solved. It seems that the Error Suspicion episodes were a better cue to errors made than subjects' explicit judgements about the correctness of their solution.

More specifically, almost all of the Direct Detection episodes and about two-thirds of the Error Suspicion episodes were triggered by erroneous solution parts. On the other hand, only 12% of the Routine Check episodes were triggered in connection with some erroneous solution part. This result, together with the finding that only five of the 326 errors analyzed were detected in Routine Check episodes suggests that such checks may not be worthwhile as long as they are only performed on the odd occasion.

As noted, two-thirds of the Error Suspicion episodes were connected to errors. It appears that the subjects did not always take optimal advantage of the information available in the Error Suspicion episodes since 27% of the errors connected to these episodes were not discovered.

Subjects had particular difficulties in detecting their solution method errors. Nearly half (48%) of these errors were not detected as compared to only 13% of the execution errors. This difficulty seems to stem, at least partly, from subjects' insensitivity to the effects of this type of error. Thus 30% of the solution method errors were not even connected to some reflective episode, as compared to 7% of the execution errors. Another cause of subjects' difficulty with solution method errors seems to be that subjects do not activate relevant knowledge when dealing with this type of error. The data show that 41% of the solution method errors connected to Error Suspicion episodes were not detected, compared to 20% of the corresponding execution errors and also that, as noted above, erroneous choice of solution method occurred more often when subjects did not put forward any arguments for their choice.

Generally speaking, the results of this study provide evidence for two types of spontaneous error detection. The detections of the Direct Discovery episodes seemed to occur in one step and may be due to a match between an explicit representation of a specific error in subjects' memory and a perceived occurrence of the same error in the performed solution. In contrast, the second type of error detection is a multi-step process. Initially, a mismatch occurs between some feature in the solution (not necessarily an error) and some expectation held by the solver. The mismatch generates a symptom that is negatively experienced by the solver. Next the solver reacts in some way to the symptom, for example by performing further diagnosis or simply by just ignoring the symptom and continuing the solution. The data show that subjects often ignored experiences of symptoms even when the Error Suspicion episode was connected to an error. An important feature of the second type of error detection is that it is a general mechanism that can detect also those errors of which the solver has no explicit representation. Parenthetically we note that this type of error detection bears similarities to what Kuhn (1962) has described as the occurence of "anomalies" in the development of science. The results above support the taxonomy of error detections shown in Figure 1.

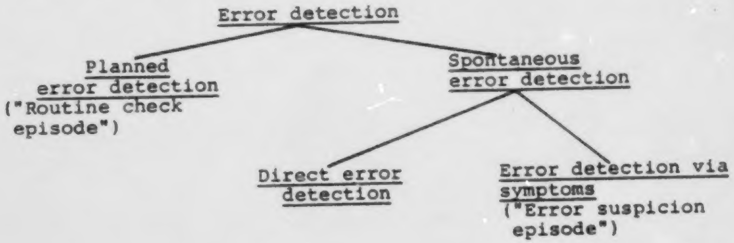
Error detection
Planned error detection
("Routine check episode")
Spontaneous error detection
Direct error detection
Error detection via symptoms
("Error suspicion episode")

Figure 1. Taxonomy of types of error detections.

Several aspects of subjects' error detection behavior were related to problem solving proficiency. Problem solving proficiency was positively correlated with the proportion of detected errors. This result held for both executive errors and conceptual errors (i.e. solution method errors and high level arithmetic errors combined). It also held for error detection in the Direct Detection episodes and error detection in the Error Suspicion episodes analyzed separately. According to the pattern of correlations, this result seems to be due to differences in the early, possibly preconscious, parts of the error detection process, rather than differences in what subjects did when a reflective episode had been triggered.

The data also showed that the sooner a relevant Error Suspicion episode occurred after an error, the greater the probability was that the error would be detected. However, there was no significant correlation between problem solving proficiency and how soon a relevant Error Suspicion episode occurred after an error.

In conclusion, the results in this study showed that subjects spontaneously detected errors either through a one-step direct detection process or through a more extended and variable process initiated by a negatively experienced symptom. The data also showed that most of the subjects' errors were connected to reflective episodes although two-thirds of the undetected errors were not. Finally, good problem solvers detected a higher proportion of their errors, probably due to some advantage in the early preconscious stages of the error detection process.

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DEVELOPING A TEACHING MODULE IN BEGINNING ALGEBRA

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ABSTRACT:

Le projet 'Strategies and Errors in Secondary Mathematics' (SESM) est en train de construire une séquence didactique en algèbre élémentaire après avoir identifier et analyser des erreurs particulières que les élèves (âgés de 12 à 15 ans) faisant dans ce sujet. Ces 'erreurs particulières' sont celles qu'a révélées le projet 'Concepts in Secondary Mathematics and Science' (CSMS); la méthodologie utilisée dans la recherche est l'entrevue clinique et l'expérimentation didactique; et les résultats indiquent que les élèves ont des problèmes avec la notion de variable, la façon de résoudre des problèmes algébriques, l'usage de parenthèses, et l'acceptation de la forme indeterminée surtout telle que $n + 3$, $2a + 5b$, etc. Pour aider les élèves à résoudre ces difficultés, nous avons constitué une séquence didactique qui se rapporte à programmer une 'machine mathématique' (cf. ordinateur). Les résultats premiers indiquent qu'on peut commencer à résoudre ces problèmes de cette façon.

The Strategies and Errors in Secondary Mathematics (SESM) project is a project funded by the Social Science Research Council and based at Chelsea College. This project, which follows on from the work of the Mathematics section of the Concepts in Secondary Mathematics and Science (CSMS) programme, commenced in 1980 and aims to investigate particular errors identified by the CSMS programme and to develop teaching modules designed to help children avoid making the identified errors.

Preliminary work on generalised arithmetic by SESM reported earlier (Booth, 1981a, 1982a, b) suggested that the errors under study (see Figure 1 for examples) appeared to relate to four areas of difficulty. Subsequent investigations involving clinical interviews, paper-and-pencil testing, and small-group teaching experiments have made it possible to describe these areas of difficulty more fully. As a result of this work, a teaching module has been designed which aims to help children avoid these difficulties. Preliminary trials of this teaching module suggest that an improvement in performance on items pertinent to the errors in question can be obtained, and that this gain in performance continues over the following two months, as shown by delayed post-test results.

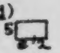
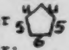
CSMS Item (abridged)	'Error' Answer	Percentage Giving Answer (13 yr.olds)
1. Area of: 	5e2, e10, 10e, e+10	41.7
2. Perimeter 	2u556, 2u16	46.3
3. Perimeter: (n sides of length 2)	32 to 42	25.4
4. Add 4 onto 3n	3n4, 7n 7, 12	44.7 17.3
5. L+M+N=L+P+N True always/ never/sometimes when	Never	55.5

Figure [illegible].
Examples of errors in generalised arithmetic identified by CSMS

A. Identifying the Difficulties. A total of 72 individual interviews with children aged 12 to 16 years from the 'middle ability' mathematics groups of five Greater London schools were conducted. In addition, 9 children from the top mathematics group of a selective school were also interviewed. Following this, the findings from the interviews were tested out by means of a teaching experiment involving three groups of 5-6 children aged 13 and 14 years in two schools, and by means of a short paper-and-pencil test designed to assess the extent of two of the identified problems, and which was administered to approximately 1000 children in three schools. As a result of this investigation, the four areas of difficulty summarised below were elaborated:

1. Interpretation of letters. Children often do not understand that letters are representing numbers, and that the number represented may be a unique value (as in $x + 2 = 5$), or an infinite range of values (as in $x + y = y + x$) (See Collis, 1975a, b; Harper, 1980, 1981; Küchemann, 1978, 1981a, b; Wagner, 1979; 1981a, b for an extensive treatment of this issue). Children may cope with this problem by ignoring the letter completely, by substituting a particular value for the letter, or by treating letters as objects which can be merely collected up (Küchemann, op.cit.). At a more sophisticated level, error may occur because the child fails to take into account the fact that a given letter may represent a range of values rather than a single unknown.

2. Formalization and symbolisation of method. Children often do not symbolise the methods they use to solve problems in arithmetic. Consequently they have difficulty in producing a generalised form of that method, as is often required in algebra. This difficulty appears to stem from three sources:

(i) Children often use non-formal 'common-sense' methods (e.g. Erlwanger, 1975; Ginsburg, 1977; Booth, 1981b; Hart, 1981); these methods can be successfully applied to easy arithmetic problems, but are rarely symbolised by the child and indeed often do not lend themselves readily to concise mathematical representation.

(ii) Even where the child uses a formal (taught) procedure, he/she may not symbolise it appropriately (i.e. in a form appropriate to algebraic representation).

(iii) Even where a child uses a formal method and symbolises it correctly, he/she may not see that this is an appropriate thing to do.

3. Conjoining in algebraic addition. Perhaps because children do not recognise an expression such as n + 3 as a legitimate answer, but rather as a sum which still needs doing, they attempt to perform the addition, giving n3 or 3n as the answer (Davies, 1978; Matz, 1980).

4. Use of brackets. Children see no need for brackets and consequently do not use them (Kieran, 1979). Children's resistance to using brackets appears to be based upon the belief that (Booth, 1982c):

(i) operations are to be performed in the order written

(ii) context overrides the above rule, i.e. the particular problem will determine which operation should be done first regardless of the order in which the operations are recorded.

(iii) the order is in any case irrelevant since the same answer will be obtained regardless of which operation is performed first.

B. Developing a Teaching Module. Based upon the above findings, a teaching module was designed with the aim of improving secondary-school children's understanding of early algebra by specifically addressing the areas of difficulty so identified. This module was set within the context of a 'mathematics machine' (Figure 2) which was to be 'programmed' to solve given problems.

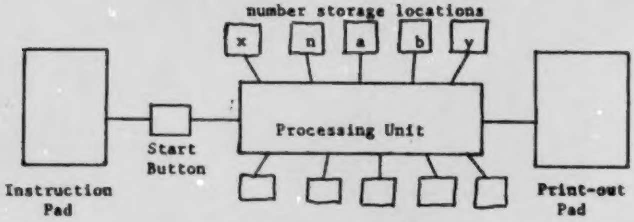


Figure 2

The use of such a model enables the above areas of difficulty to be handled by:

1. the introduction of letters at the generalised number level, by which a given letter is conceived of as having the potentiality to represent a range of values
2. concentration on recognising the formal method required to solve a problem, on the representation of that method, and on a consideration of the legitimacy and status of indeterminate or general answers as well as the meanings which might be ascribed to them
3. requiring all operations to be initially written in full (e.g. 3 x n rather than 3n), and the eventual replacement of 3 x n by 3n to be accompanied by a repeated comparison with the summed expression 3 + n.
4. overriding the order-of-operations conventions and initiating the notion of need for brackets whenever the outcome of a series of operations is non-unique.

This teaching module, which covered a period of six 35-minute lessons given over 5 to 6 days, was presented by the experimenter to three groups of 10-16 children aged 13, 14 and 15 years from mathematics groups of varying ability who had already been identified, from their performance on 21 items of the CSMS Algebra test given as a pre-test, as making the errors under study. In addition, a group of fourteen 'mixed ability' 12-year-old students who had received no previous formal teaching in algebra also undertook the programme. A parallel version of the CSMS-based pre-test was given as immediate post-test, and the pre-test version was administered as a delayed post-test two months after the immediate post-test. The results in terms of total test scores obtained for the group of sixteen 13-year-olds are shown in Figure 3; similar results were obtained for the other groups.

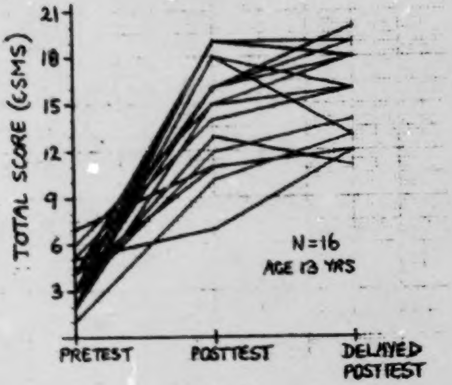


Figure 3

In addition, changes in performance on items grouped according to the area of difficulty to which they relate provides some interesting information. This analysis is currently being undertaken, and it is hoped that the insights which it may provide will enable further implications for teaching to be drawn.

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J. DISCOVERY LEARNING

HOW DO CHILDREN DISCOVER STRATEGIES (AT THE AGE OF 7) ?

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Introduction

The aim of this research is double : firstly we want to study the behaviours and strategies adopted by young children (7 to 8 years old) who discover the regularities of their environment; and secondly we want to study the possibility to favour the development of logico-mathematical structures in the child through the use of non-verbal communication devices (LOWENTHAL, 1980).

We show here that 2nd graders (7 year olds) are able to discover, to explicitly state and to explain to others the basic principle of the concept "finite automata" : the "next state - next output" function.

Procedure

We let children build railway networks (mazes) using the material conceived by COHORS-FRESENBORG (1978) : this is our non-verbal communication device. We also presented diagrams representing networks to the children and asked them to reproduce the diagrams using the concrete material, and then to analyze the functioning of the mazes.

a) Material

We described the basic bricks in a previous paper (LOWENTHAL and MARCQ, 1980). They constitute the basic elements needed to create a railway network : straight rails, curves, by-passes and switches. This material has built-in constraints which purposely restrict the number and the kind of combinations a child can make with the pieces : the teacher does not need to tell the child that there are restrictions, the built-in constraints automatically impose them.

b) Method

We chose to let the children work by groups of 2 or 3 in order to favour their verbal and social development. The groups had opportunities to compare and discuss their results.

We asked each group to reproduce a diagram of a network, building a maze and using the concrete material. The diagrams we presented became more and more complex. There were 14 diagrams. When a group had built a maze we asked these children : "What is the use of this maze ?" Each child could discover it by letting numbered trains go in good order through the network and by observing towards which exit the network would distribute these numbers : the maze must be viewed as a typical vending machine. This implied that the children had to keep track of what they were doing. We suggested a first technique : write a list of numbers on a card (0,1,2,3,...) and put this list at the entrance of the maze; put also an empty card at each exit and write on it, as soon as they leave the maze, the train number coming through that exit. Later, we suggested to keep track of all details by using a table with 3 columns : the first one for the train number, the second for the position of the switches before the train went through the maze and the last one to note the exit the train took. The position of the switches after the train went through the maze can be read on the next line, which is foreseen for the next train : the position of the switches after train n leaves the maze is the same as the position before train n+1 enters the maze.

We asked the child to make (and check) several predictions based on his maze. We also wondered whether he would produce, by generalization, a formula enabling him to make easily predictions.

Results

We observed different levels of understanding, but the children did not all go through each level, some stopped their progression before reaching the goal, others jumped ahead. We describe here the main levels.

a) All children started to use a trial and error procedure and some never went any further.

b) Soon other children claimed that there was "a regularity in the succession of exits". They then said that there was a "rule governing the network" and that the rule laid solely in the succession of exits. They did not mention then the main element : the position of the switches before and after the train went through the maze.

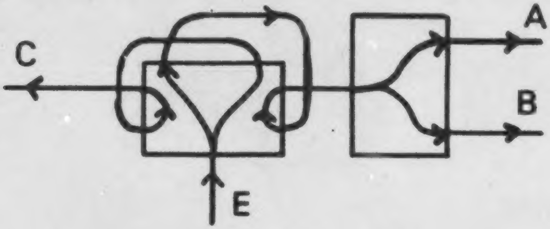

Figure 1.

The maze shown in figure 1 would thus be described by the following succession of exits : A C B C A C B C A C ...

c) Some children started to predict exits by using the mental image they had memorized for each diagram. This became impossible when the diagrams became more complex and espacially when loops were introduced, as shown in figure 2.

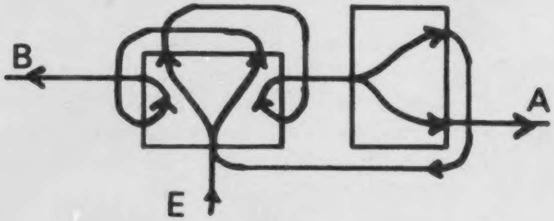

Figure 2.

Other children started to look at the table they used "to keep track". They would predict : "If the position of the switches is X, then the exit is Y" or "The exit Y can only be reached when the switches are in position X_1 or X_2 or ... X_n, and then it will always be exit Y." But these children did not say that each train determines the position of the switches for the next train, and thus the next exit : in fact everything is determined by the position of the switches before the first train enters the maze.

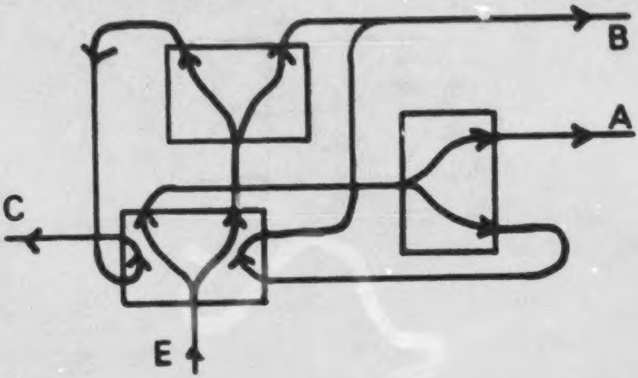

Figure 3.

d) We then produced diagrams implying apparent irregularities : in fact the first two positions of the switches, for the diagram shown in figure 3, can only be reached once (at the start) and never again ! A classical loop of length 4 starts immediately after this, which produces the following list of exits : AB CABB CABB CABB C...

The children had to confront their (incomplete) concept of finite automaton with these experimental data. Those who still used trial and error did not react. The children who based the rule upon :"the beginning of the sequence of exits" were puzzled; some tried to check the result by starting everything over again (controlling the experimental data); most tried to find a new "rule governing the network". The children who had started predicting by using a table for notations, without realizing the importance of the order of succession, discovered then that this order is important : "The first B is not the same as the second B" said a little girl, meaning : when we reach exit B for the second time, the position of the switches is not the same as for the first time. These children start reasoning about the notations they use and detach themselves from the maze as such.

e) Eventually some children realize that after a finite (initial) number of trains, they go through "the same exit" (as previously) "for the same reason" (same position of the switches) and that it is then useless to let more trains go through the maze : one simply has to reproduce the notes which have already been written down (i.e. the finite automaton is now in a loop).

f) It must be mentioned that at a certain level of complexity the children are not able to find their way through the diagram without the help of the concrete material.

To summarize, we mention the following data : out of 20 second graders, 17 are able to build a maze corresponding to a diagram; the 3 remaining ones do not seem to progress normally at school and their teacher will not allow them to enter 3rd grade with the other ones. It might be the case that a failure at building mazes could be used to detect potential learning disabilities.

As the children worked in teams, we cannot exactly say "who did what", but 11 (or 12) children were able to say that the succession of exits is governed by a rule, they were able to make some predictions but they did not always correctly describe the rule. Some were influenced by the rythm of the exits. 6 (or 7) children (out of 20) were able to state the rule and explain to others that "the rule is given by a finite (initial) part of the table for notations".

Discussion

These results show that 7 year olds are able to make experiments and test variables : they are able to discover the causes of apparent irregularities and to generalize their conclusions when they dispose of a non-verbal communication device adapted to the task. In such a context, they are also able to correct and complete their own incomplete notions in the process of concept formation : the non-verbal communication device (here the network) serves as concrete checking device.

Some 2nd graders (7 to 8 years old) proved able to discover a concept of finite automaton through the use of a communication device with built-in constraints. Some were even able to (partially) reach the level of formal reasoning : they detached themselves from the mazes, started to rely mostly on the notation system which, they had been laid to invent. They also started to instruct their fellow students to do as they did.

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K. SOCIAL ASPECTS

L'ENSEIGNEMENT DES MATHEMATIQUES N'EST PAS NEUTRE

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ABSTRACT: MATHEMATICS TEACHING IS NOT NEUTRAL.

Mathematics are a universal science; mathematical objects are abstract ones, devoid of nationality, race, religion, sex or social class. However we will see that the dressing-up of "mathematical" problems proposed to pupils are vehicular for society images which are quite representative of the leading social forces.

The study is based upon french textbooks from various times. It intends to be a contribution to the research of roots of social inequality for successes (or failures) in mathematics.

Les mathématiques sont une science universelle; les objets mathématiques sont des êtres abstraits sans nationalité, race, religion, sexe ni classe sociale. Et pourtant, nous verrons que les habillages des "mathématiques" proposés aux jeunes français à l'école sont les véhicules d'images de la société qui sont très représentatives des forces sociales dominantes et ne peuvent qu'aggraver les processus d'exclusion de la sphère des élus.

ETUDE D'UN MANUEL D'ARITHMETIQUE DE 1860:

Le manuel intitulé "Simples notions d'arithmétique théorique et pratique avec plus(1) de 600 exercices et problèmes à l'usage des écoles primaires et des écoles d'adultes" par P. D. POUJOL, instituteur, a été publié (2) en 1860 puis réédité chaque fois à 500 exemplaires. Il a atteint une onzième édition en 1898 et les plombs ont été détruits en 1908.

Dans l' "avertissement" introduisant à l'ouvrage, l'auteur cite le règlement des écoles publiques: "*L'enseignement du calcul sera dégagé de toute théorie trop abstraite. Le maître se bornera aux principes indispensables pour la pratique des opérations et s'attachera à faire résoudre beaucoup de problèmes relatifs à des questions usuelles et au système décimal des poids et mesures*".

Il poursuit en ces termes: "*Ces prescriptions si sages devaient nous servir de guide; nous ne les avons jamais perdues de vue en composant ces Premières notions d'arithmétique théorique et pratique. Nous n'avons, en conséquence, donné que les définitions, les règles, les principes réellement utiles, que nous nous sommes efforcé d'exposer dans un langage à la portée de l'enfance. La partie théorique étant ainsi restreinte, nous avons pu développer davantage la partie pratique. Aussi notre travail présente-t-il 600 problèmes, pour la plupart semblables à ceux que l'ouvrier, l'artisan, le cultivateur, le fermier, le propriétaire ont à résoudre journellement.*

Comme toutes les parties de l'enseignement doivent concourir à l'éducation morale des enfants, nous avons compris dans notre volume quelques questions propres à inspirer le goût du travail, de l'ordre, de l'économie, etc..."

Dans la "note sur la seconde édition", on peut lire:

..." *nous avons ajouté à l'arithmétique appliquée un paragraphe relatif aux rentes sur l'Etat. Il nous a semblé utile d'attirer sur ce sujet, dont l'actualité augmente de jour en jour, l'attention des élèves les plus avancés de nos écoles primaires*".

L'ouvrage, dont on trouvera la table des matières en annexe 1, est rédigé systématiquement de la manière suivante: chaque chapitre comporte des informations, définitions, "principes", "règles", éventuellement "démonstrations" purement théoriques dont nous ne parlerons pas ici, suivis de listes d'"exercices" puis de "problèmes". Après certains groupes de chapitres, on trouve des "problèmes mêlés" et des "problèmes de récapitulation". Quelquefois le premier problème de la série est donné en "exemple" et sa solution est traitée.

Voici le problème n°1: *Un père de famille a dépensé dans une année, pour son entretien, celui de sa femme et de ses enfants, les sommes suivantes: pour le pain, 259 frans; pour les autres aliments, 309 frans; pour l'habillement , 121 francs; pour la rétribution scolaire des enfants, 42 frans; pour contributions (cote personnelle et mobilière, journées de prestation), 8 francs; pour aumônes, 16 francs; et enfin pour objets divers, 28 francs. Quelle est sa dépense totale pendant cette année?*

Cet énoncé évoque la famille, l'école, des achats alimentaires et d'habillement, des impôts et des aumônes - Il sera classé (F, Sc, A, H, I, Au).

On trouvera en annexe le nombre d'occurences de chacun des thèmes socioculturels relevés. Pour expliquer le mode de détermination des critères, citons quelques exemples avec les codages attribués:

-Problème n°25: *Un père de famille avait l'habitude déplorable d'aller tous les soirs au cabaret et laissait souvent sa famille sans pain à la maison. Pendant quatre ans qu'il a mené cette vie, il a dépensé, la première année, 97 francs; la seconde 104 francs; la troisième, 112 francs; et la quatrième, 129 francs. Combien de francs ce malheureux père aurait-il épargnés, s'il n'avait pas eu cette affreuse passion de la boisson?* (Codé: Famille-Morale-Alcool-Epargne).

-Problème n°94: *Une personne laisse, en mourant, une fortune de 48560 francs; elle lègue 40670 francs à ses parents et le surplus au bureau de bienfaisance. Quelle est la part du bureau de bienfaisance dans cette succession?* (Codé: Héritage-Aumône-Famille).

-Problème n°114: *C'est en 1760 qu'aux environs de Clostercamp, en Westphalie, le chevalier d'Assas, natif du Vigan, capitaine dans le régiment d'Auvergne, mourut héroïquement pour le salut de l'armée française(1). En commémoration de son beau dévouement, une statue du héros a été placée, en 1830, sur la place publique de sa ville natale. Quel temps s'est-il écoulé depuis sa mort jusqu'à l'érection de sa statue?*

(1) *Toute la France connaît ce cri du plus sublime patriotisme:"A moi, Auvergne, ce sont les ennemis!"* (Codé: Histoire-Morale).

-Problème n°113: *Napoléon III est né à Paris le 20 avril 1808. Quel est l'âge de S. M. Impériale?*

En consultant le livre du maître, dit "Solutions des problèmes contenus dans les simples notions.....", on trouve la réponse: *113 - Sa Majesté Impériale Napoléon III a 52 ans.*

Cet exercice a été codé "Histoire - Pe", "Pe" indexant tous les cas où une "personnalisation" impose de mettre en jeu le lecteur pour pouvoir résoudre le problème; tous les cas relevés sont liés à l'histoire.

Par contre, j'ai noté "Pe_Je" les cas de personnalisation tels que:

-Problème n°312: *Je possède une petite pièce de terre complantée en mûriers oliviers et ceps de vigne. J'y ai récolté 3 quint. métr. de feuilles de mûrier que j'ai vendues à raison de 10 fr. le quintal; des raisins qui m'ont fourni 2 hectol. de vin valant 25 fr. l'hectol., et des olives qui m'ont fourni 2 décalit. d'huile valant 17 fr. le décal. Tous les frais de culture de cette pièce de terre se sont élevés à la somme de 28 fr. Quel est son revenu net?* (Codé: PeJe-Propriétaire-Cultures).

Lorsque l'énoncé parle de personnages (non historiques) désignés nommément, il est codé "individualisation", comme dans les deux problèmes successifs ci-dessous:

-Problème n° 578: *Adolphe est un garçon économe et prévoyant. Chaque semaine, il place 1 fr.25 à la caisse d'épargne. Combien y place-t-il par an?* (Codé: Ind-Morale-Epargne).

-Probème n° 579: *Ernestine est une jeune fille qui aime les pauvres. Pendant ses récréations, elle tricote des bas, brode des cols, des bonnets, ect., et gagne de cette manière 0 fr. 90 par semaine qu'elle donne à ses amis, car c'est ainsi qu'elle appelle les malheureux. Combien leur donne-t-elle par an?* (Codé: Ind-Aumône).

Les "exercices" de calcul sont de simples opérations à effectuer; il y en a une vingtaine après chaque chapitre, codés "C" (calcul sans habillage) ainsi que quelques-uns des problèmes. Mais, on peut noter que presque tous les "C" qui sont qualifiés de "problèmes" conduisent à concrétiser d'une certaine manière les nombres, à les traiter de manière dynamique comme:

-Problème n° 342: *Rendez 7 fois plus petit le nombre 242, 06.*

-Problème n° 105: *Quel est le nombre qui deviendrait 35489 si l'on y ajoutait 20984?*

Mais, à part 128 problèmes ou exercices de type C, tous les autres sont basés sur l'évocation d'un certain contexte. Les plus dépouillés sont les "exercices" de conversions et les problèmes (codés Arg) qui font allusion à l'argent seulement sans roman autour, comme:

-Problème n° 487: *Partagez 72 fr. entre 2 personnes, de manière que l'une ait le double de l'autre.*

Tous les autres nous permettent d'entrevoir une société rurale peuplée de fermiers, propriétaires, de commerçants et maquignons qui achètent et vendent du bétail, qui cultivent des céréales, des châtaignes élèvent des vers à soie, des cocons, vendent de la toile, du drap et de la soie (3). On y mange et quand on y boit c'est surtout du vin, quelquefois jusqu'à l'alcoolisme et, dans ce cas, il est toujours rappelé que ce vice (comme l'usage du tabac) nuit à l'épargne qui est la grande vertu évoquée 72 fois, bien qu'elle soit en contradiction avec les aumônes qui semblent recommandées (surtout pratiquées par des personnes du sexe féminin). Dans ce monde, on voyage assez peu (8 fois et il y a 1 locomotive). Les ouvriers et employés sont rares, comme les artisans(qui s'associent ce qui fournit des problèmes au chapitre sur les "Règles de société").

Les salaires sont cohérents (dans les divers exercices le salaire de l'instituteur reste le même). La famille est évoquée au travers des pères de famille et de l'héritage mais c'est avant tout un monde d'adultes (même dans les cas notés "Ind"). L'enfant n'est interpellé que pour se rappeler la date, pour la comparer à un évènement historique (Hist-Pe) ou pour agir sur des nombres (C). Le monde de l'école (Sc)n'est évoqué qu'au travers de l'instituteur ou de problèmes tels que:

Problème n°127: *Un élève avait à copier 48 pages d'histoire; il en a copié une première fois 12 pages, et, une seconde fois 17. Combien lui en reste-t-il?* (Codé: Sc).

Mais, pendant le cours de mathématiques, l'enfant va apprendre à devenir un bon français moyen: cultivateur, propriétaire, père de famille, économe, sobre, bon citoyen connaissant bien les grands anciens qui lui sont rappelés (grands écrivains, hommes de science et de guerre), comme dans:

Problème n° 263: *Fénelon, l'immortel auteur du Télémaque, naquit dans le Quercy en 1651 et mourut en 1715. A quel âge est-il mort?* (Codé: Histoire-Littérature).

Une étude analytique est faite sur divers manuels français du début du XXè siècle par A. Harlé (Mémoire de D.E.A. en cours à l'Université Paris VII): on retrouve longtemps cette conception, s'agit-il de ce que Hans Freudenthal appelle "Mathematics as an educational task"? ou plutôt de Mathématiques comme moyen d'intégration sociale? Sans aller chercher hors de France des cas limites, souvent cités, d'exacerbation du nationalisme dans les écoles, on trouvera en annexe 3 une illustration extraite d'un manuel français de 1947 où nettement les allemands, japonais et italiens en uniformes et à allure martiale sont opposés aux paisibles dandys français et aux fumeurs de pipe britanniques.

Après la réforme des "Mathématiques-modernes":

Les manuels français contemporains ne sont pas seulement ceux d'une société plus urbaine que rurale et où l'on dispose d'automobiles et de téléviseurs; la société représentée y est toujours celle d'une classe moyenne aisée. Dès 1973, Cl. LIGNY dénonçait, dans un article des Temps Modernes intitulé "Mathématiques innocentes", une liste de "situations familières et concrètes" piquées dans deux manuels de 6è et 5è (4).

Loin d'avoir diminué dans un enseignement que certains ont qualifiés de "Bourbakiste" et "d'effroyablement abstrait", l'omniprésence de la société modèle dans les manuels de mathématiques est beaucoup plus dense qu'autrefois. La première phrase des programmes et objectifs officiels de mathématiques est: "A tous les niveaux de l'école primaire, il importe de partir de situations tirées du vécu de l'enfant, liées à ses intérêts spontanés et provoqués, et de les exploiter collectivement et individuellement dans le cadre de la vie de la classe".

Désormais la société n'apparaît plus seulement dans les problèmes, chaque chapitre de cours doit commencer par une petite histoire (le plus souvent accompagnée d'images ou de photos dont l'analyse est elle aussi très intéressante tant sur le plan socio-culturel que sur le plan de l'adéquation au concept mathématique qu'elle censée évoquer) et c'est de cette histoire que sont "extraites" les notions mathématiques présentées.

La différence est donc fondamentale:

- Dans le premier cas, on enseignait les mathématiques pour les appliquer, les utiliser en vue de devenir un bon citoyen.
- Dans le deuxième cas, la bonne intégration à la société environnante permet d'accéder au monde des mathématiques.

La première démarche était celle de l'époque optimiste où l'école était vue comme outil de promotion sociale, la deuxième est celle de l'ère de la "reproduction" au sens de Bourdieu: Quiconque est étranger à un certain environnement socio-culturel a peu de chances d'y pénétrer et dénoncer le rôle de l'enseignement des mathématiques dans la sélection par l'échec scolaire est devenu un lieu commun.

Pourtant une autre différence frappe: il s'agit beaucoup moins d'une société d'adultes, il est question d'enfants, de la classe, des vacances, des cadeaux reçus pendant les fêtes ce n'est plus le monde des adultes nantis auquel on cherche à accéder plus tard, c'est le monde des enfants gâtés auquel on rêve d'appartenir. Pour faciliter le rêve, la personnalisation est maintenant relative à l'élève lui-même: les auteurs de manuels ne disent presque plus "je" mais ils disent très souvent "tu". Plus même, au lieu de dire "ta mère" ou "ta maman", ils disent "Maman", "Papa", Grand-mère", comme lorsque les grandes personnes s'adressent à de jeunes enfants en bêtifiant; les élèves ne doivent pas se laisser piéger par cette démagogie et tel qui ne sentirait pas que "Maman" dans un problème n'est qu'un symbole de variable abstraite et intégrerait son vécu réel et son affectivité, aboutirait nécessairement à un échec (5).

Tout se passe comme si les habillages par des situations extra-mathématiques étaient des procédés de marketing destinés à "faire vendre" des notions mathématiques tout comme les emballages à la mode pour les jeux par exemple:

Pourquoi cet excellent exercice intellectuel qu'est le Master Mind qui passionne les enfants (même parmi les non-brillants-scolairement a-t-il besoin de de ces images représentant sur la boîte des cadres-supérieurs-dynamiques)? (6).
Que les mythes guerriers qui transparaissent dans le traditionnel jeu d'échecs se retrouvent dans les war-games en vogue n'est pas pour nous étonner mais on frémit tout de même en découvrant le nouveau jeu "After the holocaust"!

NOTES:

(1) En fait, il y a 624 exercices et problèmes.

(2) Par la Maison d'Editions Dezobry, Magdeleine et Cie pour les premières éditions puis repris par Charles Delagrave et Cie.

(3) Une descendante de l'auteur a écrit à ce sujet "l'Ethnographie par les manuels scolaires - Ethique et arithmétique cévenoles aux XIXè siècle" - par Geneviève Poujol - in Réforme - 23 février 1974.

(4) Voir aussi plusieurs exemples dans J. Adda: "Difficultés liés à la présentation des questions mathématiques" (E.S.M. 7-1976).

(5) Cf. J. Adda - "Quelques aspects de la relation aux mathématiques chez des enfants en situation d'échec scolaire dans l'enseignement élémentaire" in Actes du IIè Colloque "Langage et Acquisition du Langage" - Mons-septembre 1980.

(6) Selon le prix du jeu, le vêtement est plus ou moins "sport". Les modèles de luxe sont décorés d'hommes aux tempes grisonnantes accompagnés de jeunes femmes, et, pour le plus cher, ils sont tous deux en tenue de soirée et c'est une jeune asiatique qui se penche sur l'homme blanc.

ANNEXE 1

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-Fermier, cultures, élevage: 85
-Propriétaire, propriété: 41
-Commerçant, négociant, marchand, maquignon 125
-Ouvrier, employé: 39
-Artisan: 18
-Instituteur: 8
-Ancien militaire pensionné: 1
-Banquier: 2
-Achats(entretien, aliments, meubles): 6
-Aumônes: 9
-Salaires: 33
-Epargne: 82
-Héritages: 2
-Dettes: 14
-Impôts: 1
-Sc: Vie scolaire 12

-Voyageurs, voyages: 8
-Locomotive: 1
-Maison (carrelages): 2
-Education civique: 8
-Morale: 8
-Vin 20 dont 2 Alcoolisme.
-Tabac: 1
-H: Habillement (sans artisan) 2.
-Ages (autres que pour personnages historiques): 3
-Histoire: 23
-Géographie: 3
-Calendrier et calcul du temps: 8
-Lecture(nombre de pages de livres): 2
-Littérature (avec histoire): 8
-Bassins, fontaines, contenances: 16
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-Personnalisation Pe (Histoire): 11
-Individualisation (Gustave, Paul,...): 13

.../...

Annexe 2.

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Annexe 3 -(Mille 42-1947)

III. — La représentation graphique.

116. Un dessin parle aux yeux mieux que des chiffres. — Aussi se sert-on de dessins quand on veut frapper l'imagination par la comparaison de nombres. Les affiches, les ouvrages de vulgarisation en font un emploi régulier.

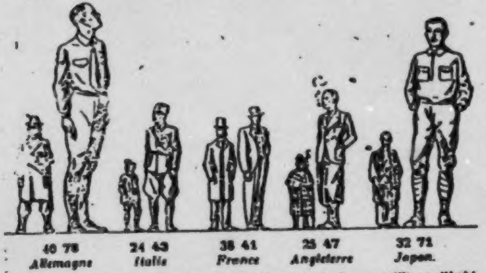

Fig. 21. — Comparaison des populations exprimées en millions d'habitants, de 5 nations, sur leurs territoires actuels, en 1865 et en 1930.

Assessment of Mathematics Anxiety - An Update

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This is the report of a research study of possible causes of mathematics anxiety by the author and co-investigator Eugene E. Levitt, Department of Psychiatry, Indiana University of Medicine (Levitt, In press).

In this investigation we have examined four correlates and potential causes of mathematics anxiety: general trait anxiety, test anxiety, gender and mathematics sophistication. We have found all four to be positively correlated.

The subjects in this investigation were 229 female and 239 male students in sections of six different undergraduate mathematics courses. All courses are considered to be at the first or second year university level but students in the various courses represent a wide range of mathematics sophistication. These students also represent a broad range of career goals. The six courses ranged from a broad based remedial or beginning algebra course to an algebra and trigonometry course for technicians to a calculus course for science majors.

For this report I have numbered the courses 1 through 6. A brief description of each course follows.

1. A broad based remedial algebra course designed to correct a deficiency in mathematics background. This is an introductory level algebra course.
2. Another broad based remedial algebra course designed also to correct a deficiency in mathematics background. Topics in this course that are not included in course #1 are the quadratic formula and logarithms. Many of the students in this course plan to be business majors.
3. This is a general mathematics course designed specifically for elementary teaching majors. The topics are an extension of those in the first through eighth year mathematics. The students are predominantly female with a varied mathematics background with many weak students.
4. This course is the first semester of a two-semester sequence of algebra and trigonometry for technology. The students in this course are predominantly males with career goals in technology.
5. This course is an extention of basic algebra to business and social science applications. It includes linear programming and the simplex method. There is a prerequisite of two years of high school algebra. The students are mostly business majors.

6. This is a course in differential calculus. It is the first semester in the calculus sequence for science majors. The course includes analytic geometry and has a prerequisite of two years high school algebra and one year of geometry and one year of trigonometry. The students are predominantly males majoring in science or engineering. The students in this course have the strongest mathematics backgrounds.

The students in these courses gave us a sample that represented the various levels of mathematics sophistication. This ranged from students in course #1 with very weak backgrounds to those in course #6 with very strong backgrounds.

The following table indicated the gender distribution of the sample.

Male	Female	Course	
39	65	1	Remedial Algebra (Non credit for graduation)
48	40	2	Remedial Algebra (Elective credit non-science)
0	45	3	General elementary math - elementary education
46	21	4	College algebra & trigonometry -technology
42	51	5	Finite math for business majors
54	17	6	Integral calculus & analytic geometry
239	229	Total	

Three anxiety inventories were administered to all 468 students during regular class hours. These were as follows: Test Attitude Inventory, TAI, (Spielberger, et al, 1977), the State-Trait Anxiety Inventory, STAI, (Spielberger, et al, 1970) and the Mathematics Anxiety Rating Scale, MARS, (Suinn, et al, 1972). The TAI and STAI each consist of 20 items with responses recorded on a 4-point Likert scale. The MARS has 98 items and responses are made on a 5-point Likert scale.

The following table shows the mean scores for the female and male students on the three inventories grouped by the different courses.

Mean Scores by Course and Gender

Course	TAI		STAI		MARS	
	Male	Female	Male	Female	Male	Female
1	40.4	47.8*	38.2	40.7	200.3	213.9
2	35.9	41.4*	37.6	38.1	159.0	169.3
3	--	43.5	--	38.6	--	207.6
4	41.1	43.2	36.1	38.9	185.6	186.8
5	33.2	41.9*	35.2	37.7	145.7	178.9*
6	34.6	37.3	36.0	37.5	149.7	148.3

* male-female difference is statistically significant at 5%

This information reveals a clear trend for females to score higher than males on all three instruments and at almost every course level. This agrees with findings in the literature (Betz 1978). A class by sex analyses of variance indicates the male-female differences summed over courses are significant on all three tests. But t-tests show that only the four starred inter-sex differences within levels reach the 5% level of significance. The population of women at large admits to more trait anxiety, test anxiety and mathematics anxiety than the population of men but the differences seem to be small.

Statistical analyses of the data collected indicated a positive correlation between mathematics anxiety and all four correlates: general trait anxiety, test anxiety, gender and mathematics sophistication. However, the degree of the correlations differs greatly. General trait anxiety contributes very little to the mathematics anxiety scores while test anxiety accounts for 28 to 36 percent of the mathematics anxiety. It is not suprising that test anxiety has a positive correlation to mathematics anxiety as measured by these inventories. But the correlation is not as high as one might suspect and leaves about two-thirds of the mathematics anxiety to be contributed to other causes.

As stated before, females scored higher on all three inventories, thus, test anxiety and gender are confounded. The exception to the correlation between gender and mathematics anxiety is the scores for course 6 (Calculus). The students in this course have the strongest mathematics backgrounds. The mean score for the female students in this course is the lowest for all female groups and lower than most male groups. This would indicate that female science majors with strong mathematical backgrounds are no more anxious than most male students and less anxious than students in general with weak backgrounds.

Of the four correlates investigated mathematics sophistication appears to be a powerful factor in mathematics anxiety for both male and female students.

Most attempts at remediation have looked for clues in case histories of the highly mathematics anxious students. Perhaps an in-depth clinical study of those students who score minimal levels of mathematics anxiety and have high levels of mathematics sophistication would be fruitful.

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L'ENSEIGNEMENT DES MATHEMATIQUES DANS LE TIERS-MONDE

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SUMMARY - The aim of this paper is to present

a) a few of the inconveniences for developping countries about mathematics taught in foreign institutions,

b) the benefit that such original cultures can draw from a mathematical activity,

c) the solutions which could be given to diminish the inconveniences and to increase the benefit of a mathematical activity in developping countries.

Mathematicians in developping countries are generaly isolated and can be helped by an I.I.M. (an International Institute of Mathematics) which could be created by U.N.O. for instance.

The aims of the I.I.M. would be to diminish the required knowledge to understand mathematical subjects of high level or open problems, to send a documentation adapted to each special situation of isolated mathematicians, to research new heuristic point of view on any kind of mathematics, to study interactions between mathematics and societies, etc ...

I - INTRODUCTION - Les problèmes que pose l'enseignement des mathématiques dans le tiers-monde sont trop complexes pour être abordés en quelques pages dans toute leur généralité. Nous nous limiterons au cas de l'enseignement des mathématiques dans les établissements français à l'étranger et au problème de l'accès à l'enseignement supérieur.

Comme toutes les autres activités scientifiques, l'activité mathématique a des retombées technologiques dont les effets sont parfois impressionnants et peuvent menacer l'existence de mentalités incompatibles, au moins en apparence, avec une culture mathématique. Afin de ne pas être entraîné dans des polémiques stériles qui gêneraient la recherche de remèdes possibles à l'impérialisme culturel des mathématiques, il convient de ne pas considérer les démonstrations et les calculs comme étant des instruments qui décident de la vérité ou de la fausseté des assertions. Ce rôle d'oracle ne peut qu'entraîner une réaction de rejet ou une soumission sans réserve dans un esprit étranger aux mathématiques. Il semble préférable de

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présenter les mathématiques comme étant des techniques et des langages destinés à rechercher et à décrire des voies d'accès possibles vers des buts qu'on peut choisir librement (les conclusions d'un théorème) quand on est dans une situation imposée par les circonstances (les hypothèses). Dans cette optique, on peut choisir une des solutions suggérées par le raisonnement mathématique ou courir le risque de ne pas en tenir compte. C'est avec cette dernière conception des mathématiques que nous exposerons quelques uns des inconvénients du système actuel de l'enseignement des mathématiques dans le tiers-monde, que nous montrerons l'importance du rôle que des mathématiciens du tiers-monde peuvent jouer et que nous proposerons un projet qui aiderait les mathématiciens à contribuer à l'épanouissement des cultures originales du tiers-monde.

II - EFFETS CULTURELS DE L'ENSEIGNEMENT DES MATHEMATIQUES DANS LE TIERS-MONDE -

S'il y a beaucoup à dire sur les méthodes d'enseignement des mathématiques et l'adaptation des programmes au niveau du primaire et des premières années de l'enseignement secondaire, on peut affirmer qu'au niveau du second cycle, l'enseignement français des mathématiques ne présente pas plus d'inconvénients dans les pays étrangers qu'en France. La finalité traditionnelle des programmes est d'apprendre à manipuler des équations, à construire des graphiques, à raisonner sur des figures géométriques et à inventorier les cas particuliers qui se rattachent à un problème général. Quelque soit la nature des études supérieures qui prolongeront la formation secondaire qu'aura reçu l'élève, la maîtrise de ces techniques ne peut être nuisible. De plus, l'usage du français n'entraîne pas de conflits culturels majeurs car une connaissance superficielle de la langue n'est pas un inconvénient grave . Au contraire , l'élève n'est pas troublé par les sens parasites d'un mot quand il est utilisé dans un sens purement technique. Cependant, depuis quelques années, certains ont cru déceler des arrières pensées idéologiques quand on a introduit dans les programmes la théorie des ensembles, l'algèbre linéaire (les mathématiques modernes) et, plus récemment, l'informatique.

Par contre, au niveau des universités, la forme actuelle de l'enseignement des mathématiques est manifestement de nature à étouffer l'épanouissement des cultures originales.

La finalité de cet enseignement est, en fait, de fabriquer des agrégés, des docteurs d'état , des grades universitaires (maîtres-assistants, maîtres de conférence, professeurs, etc ...), des élèves des Grandes Ecoles,etc ... Cette finalité n'est sûrement pas d'aider des mathématiciens capables de remettre en cause le point de vue des mandarins sur les mathématiques.

Cette volonté de fabriquer une hiérarchie artificielle des intelligences au

service des grands maîtres de l'université apparaît dans le choix de la méthode de présentation de certaines notions de base :
n'importe qui peut comprendre en quelques instants ce qu'est la notion de groupe à partir de l'idée de permutation et de composition de deux bijections, une topologie à partir de la caractérisation d'une famille de parties d'un ensemble, une structure d'anneau ou de module à partir de la notion simple de transport de structure. On préfère introduire ces notions à partir de l'idée de groupe abstrait, de la topologie usuelle ou de listes d'axiomes présentées de manière dogmatique. On dénigre des outils mathématiques étrangers, des domaines entiers de la mathématique sans aucune analyse ni critique sérieuse. On affirme enfin que les mathématiques, c'est une affaire de goût et d'esthétique !

Ces modes de présentation de notions fondamentales et ces opinions rendent l'étudiant esclave de la parole du maître venu de l'occident.

La conséquence de l'esprit actuel de l'enseignement supérieur est que l'étudiant du tiers-monde qui réussit brillamment ses examens est imprégné de culture occidentale et cette culture ne lui est d'aucune utilité pour enrichir et revaloriser sa propre culture.

III - CONTRIBUTION POSSIBLE DES MATHEMATIQUES A L'EPANOUISSEMENT DES PEUPLES DU TIERS-MONDE -

Les mathématiques figurent parmi les activités qui peuvent le mieux contribuer à rehausser le prestige d'une civilisation :
les mathématiques jouent un rôle central dans l'accroissement de nos connaissances scientifiques et techniques donc du pouvoir d'action de l'homme sur la nature.

Une culture qui engendre des mathématiciens ne peut qu'être représentée par des sociétés qui sont membres à part entière de la communauté scientifique internationale. Elle bénéficie du droit implicite d'être défendue par cette communauté.

De toutes les civilisations se dégagent des philosophies qui ne peuvent s'imposer que par des contraintes morales ou physiques ou être balayées si elles ne parviennent pas à définir leur rôle dans le développement actuel de nos connaissances et de notre savoir-faire.

L'histoire des sciences montre que la recherche de solutions à des problèmes soulevés par des cosmogonies, des croyances et des questions philosophiques ont été souvent de puissantes motivations à des recherches en mathématique. Réciproquement, l'activité mathématique a souvent obligé les civilisations à remettre en cause leurs fondements.

Indépendamment de son rôle dans l'évolution des sociétés et de ses retombées technologiques, l'activité mathématique a d'autres avantages.

Un texte mathématique, contrairement à un texte poétique, littéraire ou philosophique n'est pas appauvri ou dénaturé quand il est traduit dans une autre langue. Au contraire, l'effort de la traduction exige un contrôle rigoureux des raisonnements et des définitions précises qui améliorent la qualité du texte.

Si le mathématicien est souvent handicapé par son isolement, il n'est pas freiné dans son travail par l'absence de moyens matériels. Il peut se passer de laboratoire et n'a pas le sentiment d'impuissance que peuvent éprouver des physiciens, des chimistes ou des biologistes quand ils ne peuvent bénéficier des puissants moyens de l'industrie moderne.

Encourager l'activité mathématique est le moyen le moins coûteux qu'une communauté humaine puisse envisager pour faire valoir la richesse de sa pensée.

Toutefois, il ne faut pas perdre de vue que le prestige d'un mathématicien ne profite pas à sa famille d'origine mais à la société qui l'a formé, qui a inspiré ses créations et ses découvertes ou qui a su en reconnaître la valeur.

IV - PROPOSITION DESTINEE A FAVORISER DES ACTIVITES MATHEMATIQUES ORIGINALES DANS LES PAYS DU TIERS-MONDE -

Un étudiant du tiers-monde qui a conquis des diplômes ou des grades universitaires étrangers n'a souvent d'autre alternative que de retourner dans son pays et de renoncer à faire fructifier la formation qu'il a reçu ou de s'intégrer dans le contexte étranger qui l'a formé, mettre en veilleuse sa culture d'origine et devenir un professeur de mathématique ou un chercheur parmi des milliers.

La fréquence de ce genre de dilemme pourrait être diminuée par la création d'un I.I.M. (un Institut International de Mathématiques) dépendant, par exemple, de l'O.N.U. ou de l'U.N.E.S.C.O. dont le rôle serait de venir en aide aux mathématiciens isolés.

La finalité de cet I.I.M. doit être avant tout de lutter contre l'élitisme en mathématiques par une démystification des langages et des formalismes qui nécessitent parfois des années d'efforts pour être compris et maîtrisés alors que la plupart des problèmes ouverts peuvent être énoncés en des termes accessibles à des étudiants de première année d'université.

Pour être efficace, cet I.I.M. doit se fixer des missions précises :

a) prévoir des avantages honorifiques et financiers encourageant les mathématiciens à favoriser une meilleure compréhension de l'activité mathématique par un public plus large. On peut, par exemple, instituer un doctorat international sanctionnant des travaux qui

1) abaissent le niveau de formation préalable nécessaire à la compréhension d'un domaine d'activité mathématique,

2) exposent une approche heuristique nouvelle d'une notion mathématique,

3) analysent les interactions existantes ou possibles entre l'activité mathématique et la vie culturelle ou sociale.

b) organiser un service d'information permettant de communiquer aux mathématiciens isolés les renseignements dont ils estiment avoir besoin et d'entreprendre des recherches et des études visant à améliorer les sources d'information existantes et à remédier à leurs défaillances éventuelles.

Pour être crédible, un I.I.M. doit s'imposer des règles de conduite garantissant son objectivité.

Le choix de ces règles est délicat et reste un problème ouvert en raison des progrès inquiétants des techniques de propagande qui affaiblissent le sens critique des individus.

Il semble toutefois qu'un I.I.M. peut adopter sans risque la devise suivante:

" S'informer,déduire et informer sans juger "

L. NEUROPHYSIOLOGY

Table 3.1 Correlations between differences of N.S. and standard scores and dominance of right hemisphere by true/false.
n = 11

Scores difference	r	rho
1) Between total N.S. and total standard	** .608	*** .764
2) Between algebraic part of N.S. & total standard	** .524	*** .718
3) Between analitic part of N.S. & total standard	** .554	** .591
4) Between N.S. uniform continuity and standard uniform continuity	* .431	** .542

4. Paradoxes and Diagonal Processes

Preliminary experiments suggested the theory that diagonal processes and set theoretical paradoxes are related to the inhibition of the right hemisphere by the left. This inhibition is part of a cognitive process in which the right hemisphere integrates the set of all elements which have a certain property. The left hemisphere produces an additional element having the same property. The left hemisphere must overcome the "objection" of the right hemisphere to the existence of such an additional element.

This conjecture was tested in the course on Philosophy of Mathematics in the spring term of 1980.

Some of the results are in table 4.1. The conjecture is supported by the results.

Table 4.1 Correlations between total scores on questions on diagonal processes and paradoxes, and neuropsychological tests.
n = 33

Neuropsychological counting test	r	rho
1) Simultaneous counting of dots by true/false	* -.234	* -.246
2) Average of all ordinal counting by true/false	*** .493	*** .461
3) Hemispheric Dominance: Difference between (1) and (2)	*** -.562	*** -.540

5. Schools in Foundations of Mathematics

According to Kant "Geometry is based upon the pure intuition of space. Arithmetics achieves its concepts of number by the successive addition of units in time" (Prolegomana).

Frege's Logicism accepts Kant's view on Geometry and rejects his view on arithmetics. Logicsm bases Arithmetics on Logies of sets.

On the other hand Brouwer's Intuitionism accepts Kant's view on arithmetics and rejects his view on Geometry.

Space and time are related to the right and left hemispheres, respectively. Therefore we conjectured that preference of foundational schools may be related to the hemispheres.

Another way to approach Mathematics according to two foundational schools is Nominalism versus Platonism.

Nominalism accepts individuals, and therefore may be related to the left hemisphere which processes individual details of information.

Platonism accepts classes of individuals and may be related to the right hemisphere which integrates individuals into a new whole.

These conjectures were tested in the course Pyilosopy of Mathematics in the pring 1981 term. The students were asked in the examination which foundational schools they prefered. Only the 13 students which understood all the schools were included in the sample. The results are in tables 5.1, 5.2, and support the conjectures.

Table 5.1 Mann-Whitney test for 7 logicists compared with 3 intuitionists.

Neuropsychological counting test	u_1	u_2	Significance
1) Simultaneous counting of dots by true/false	21	0	***
2) Average of all ordinal counting by true/false	0	21	***
3) Hemispheric Dominance: Difference between (1) and (2)	21	0	***

Table 5.2 Mann-Whitney test for 6 platonists compared with 3 nominalists.

Neuropsychological counting test	u_1	u_2	Significance
1) Simultaneous counting of dots by true/false	16	2	**
2) Average of all ordinal counting by true/false	0	18	**
3) Hemispheric Dominance: Difference between (1) and (2)	9	9	N.S.

Notations

r	Pearson's correlation coefficient.
rho	Spearman's correlation coefficient.
u_1	Number of pairs in which students who prefered ordinal schools had lower hemispheric scores than students who prefered set theoretical schools.
u_2	Number of pairs in which students who prefered set theoretical schools had lower hemispheric scores than students who prefered ordinal schools.

N.S.	not significant	$p > .1$
*	significance of	$.1 > p > .05$
**	Significance of	$.05 > p > .01$
***	Significance of	$.01 > p$

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MATHEMATICS AND THE HEMISPHERES OF THE BRAIN *

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1. Introduction

Our main goal is to examine the conjecture that ordinal mathematical concepts are related to the left cerbral hemisphere and cardinal mathematical concepts are related to the right cerebral hemisphere.

The method of research was to find statistical correlations between scores on ordinal and set theoretical mathematical topics and hemispheric tests.

The hemispheric tests that were applied are:

1) Counting of tachistoscopically represented dots. The relation of this test to the right hemisphere has been established by independent investigations.

2) Counting of signs appearing rapidly one after another.

This test is new.

The relations between sex and handedness and the tests have been examined. These examinations support the connection of the tests to the hemispheres.

The test for the right hemisphere is reliable.

The new test for the left hemisphere is supported only by experiments conducted in our research, therefore it is less reliable.

An additional test was the counting of tachistoscopically represented different letters. This test is related to both hemispheres.

2. Arithmetics

A course about Philosophy of Mathematics was given at the Technion in the winter term of 1980. The course included Peano's ordinal Arithmetics, taught first, and Frege's cardinal Arithmetics which was taught latter. Participation in the neuropsycological tests was a requirement of this course.

Some of the results are given in table 2.1. They indicate the possibility of predicting relative success of grown up students in the two approaches to arithmetics.

---

* This work is a part of a doctoral thesis of the author, which was carried out under the supervision of Professor A. Evyatar.

The results also support the relation between the right hemisphere and cardinal arithmetical concepts and the relation between the left hemisphere and ordinal arithmetical concepts.

Table 2.1 Correlations of the difference between total scores on Frege's arithmetics and Peano's arithmetics in the test of the course and difference between standardised results of Neuropsychological tests n = 43

Substracted neuropsychological tests results	r		rho	
1) Simultaneous counting of dots by true/false and average of all ordinal counting by true/false	**	.311	**	.349
2) Simultaneous counting of letters by true/false and verage of all ordinal counting by true/false	***	.474	***	.494
3) Simultaneous counting of letters by true/false and ordinal counting of letters by true/false	***	.503	***	.531

3. Standard and Non Standard Analysis

In the winter term of 1979 the first year students in Mathematics at the Technion learned Calculus by taking two parralel courses. One course was the standard approach, while the other one was based on the non standard (Robinson) approach. The text book in the N.S. approach was Foundation of Infinite Calculus by H.J. Keisler. Almost all students volunteered to participate in the neuropsychological test. Some of the results are given in table 3.1. As can be seen, it is possible to predict relative success of the students in the two approaches to Calculus. The results also support the relation between the right hemisphere and actual infinity, and the relation between the left hemisphere and potential infinity.

Other results show that the standard approach is positively correlated to both hemispheres. The N.S. approach is positively correlated to the right hemisphere and negatively correlated to the left.

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pme 6

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Preface

This second volume of the proceedings contains summaries of some plenary lectures and a few short communications that could not be inserted in the first volume.

It also starts with a conference guide containing the conference programme, busline information, maps, information etc ...

We hope you will enjoy the conference and look forward to a fruitful scientific exchange.

Alfred Vermandel, chair & editor
Eric Schillemans, coördinator local organizing committee
Willem Kuyck, host professor RUCA, Dean of the Faculty of Sciences

We are happy to welcome you in our University and to be available for rendering your stay in Antwerp an efficient & pleasant experience.

All congressworkers

Marleen Van Barel, conference secretary
Marc Vansteenkiste, tourism, entertainment, accomodation, transport ...
Romain Pelsmaekers, Katia Pelsmaekers, Minke Zuidema & son, René De Paepe, hosting services
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Paul Clerckx, reprography

And many others ...

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II. PLENARY LECTURES

due to some last moment changes in the plenary programme only the papers of W. Kuyk and G. Papy could be inserted.

THE NEED FOR AN EPISTEMOLOGICAL UNDERSTANDING OF MATHEMATICS

Georges PAPY. (University of Brussels, BELGIUM)

I am very glad to have been invited in that strange world of psychologists interested in mathematical education. Some of my brothers-in-math are even more pleased of my presence here. I think they consider that meeting as something like a cage with leos; the latter being of course the psychologists. And they are waiting for some delight when I shall be between your teeth.

It is true that the relations between mathematicians and psychologists have always been very strange. And I think that it is of major importance to try to understand why.

Voltaire said once that if he were accused to have stolen the towers of Notre-Dame of Paris, he would first flew away. With the very same words the great french mathematician Henri Lebesgue said that when a mathematician hears speaking of pedagogy, he immediately flews away. And you may replace *pedagogy* by *psychology*, as well.

Why does e try to escape ?

Twenty years ago in Digne I was moderating a discussion of a session of the International group for the study and improvement of mathematical education.This session was very special indeed, because mathematicians were discussing about the psychological and affective factors in mathematical education. In my opinion the discussion was exceptionally fruitfull, because namely of the great sincerity and honesty of the confession of one of the participants, a very well known university professor of mathematics, who explained how deeply he had been perturbed because of the presence in his class when he was a student of an exceptionally bright and clever and quick student, who always found the solution before he could himself reach the end of the problem. We all know how a too good pupil in a class can kill the others, and our colleague explained how he could find a way and a psychological attitude permitting to finally survive. Most of the audience were deeply and favorably impressed, but I noticed at the interruptions some small groups very critical and disapprouval. At the lunch, that we were taking together, living in community, one of those opponents, in the front of all the others attacked me courageously : "When shall we finally hear about interesting things, namely mathematics ?" I realized that a malicious tumour was beginning to be a danger for the group itself, so that I had to answer the question. Did I well or not, I declared : "It is because you refuse to face your own psychological problems that you would like to escape the whole subject." I certainly did know how right I was. My words produced the complete desintegration of the poor opponent, who was not able to pronounce any word, and vanished under cries and tears, and took several hours to recover. Without being racists, all soccerplayers know that goalkeepers are of a very special psychological kind. The same way we should admit that mathematicians are generally speaking, very psychologically ... let us say exceptional. I suggest they could be a very good subject of studies for many kinds of psy (-chologists, - chanalysts, - chiatrists). But in that lecture, you will permit me to forget that more or less *ad hominem* argument.

Cn another side I believe that the relations between mathematicians and psychologists are perturberd by an important misunderstanding. On one side I perfectly know that most of the mathematicians suffer an enormous complex of superiority towards psychologists (and other ϕ,s) and psychology that they hardly consider as a science. They consider psychology cloody, verbous, idle, and not at all rigorous. On the other hand psychologists have very often a complex of inferiority towards mathematics, so difficult, abstract, complicated, and at the same time a complex of superiority towards the mathematicians too coarse to understand anything in the fine psychology. Both speak special technical languages using words of the vernacular vocabulary in a restricted and often somewhat different meaning. But each of them believe his own language is objectively much simpler, more natural and easier than the language spoken by the other. They both agree psychology is a collection of thoughts about facts, more than a mere collection of facts and that mathematics deserve more the title of science than psychology. I believe this is one of the reasons of the misunderstanding. For me it is clear that *Psychology* is a concrete science, whose objects is namely to describe and explain how the human brain is working. Perhaps is it - following Piaget - still a baby science not always sure on its legs, but a science for which a fact is more respectable than a Lord-Mayor. And this quality is certainly not shared by mathematics. The sadest short love story has for a very long time consisted in the sentence *"Two parallel lines felt in love"*, because mathematicians proclaimed *"Two parallel lines never meet each other"*. But after centuries and centuries of that statement, they decided for the happy end : *"Two parallel lines meet"* ... (at infinity, ... but nevertheless ...). The equation $x^2 + 1 = 0$ had for a long time no solution. Then it got the two imaginary solutions $\pm i$. And later an infinity of quaternions as solutions ... For Aristotle, geometry, the mathematics of that time, was *not* a science. And for Plato who wrote on his *lyccum* : "That no one enters here if he is not a geometer" geometry was a disciplin *prior* to philosophy itself. For a very long time, mathematics have been called "the geometrical art" and it is well true that by many of his doings, of his attitudes, of his aspects, the mathematician looks more like an artist than like an experimental scientist.

The more I think about mathematical activity, the more I believe it is a concert played by very distinct instruments, and this is happily not the exclusive priviledge of mathematics. A first simplification consists in looking at mathematical activity as dialectic : a kind of dialog - or sonata - between the two halves of the brain cortex. The right brain insures the global, synthetic, intuitive view in the context of the contemporary human culture. The left brain takes care of the formulation, and organizes the individual steps of the ballet. Mathematics are a social activity in the human society which is cultured-cumulative.

The intuitive views and technical instruments vary along the History. Following Topfler, the history of human civilisation is marked by three big waves, each recovers the preceding one without erasing it. The first wave was the one of agriculture and handy craft and lasted more than ten thousands years before the appearance of the second wave, the one of industry, who lasted only some centuries before the arrival of the third wave, that of informatics. The second wave permits the substitution of the engine for many works of muscles. The third wave permits the substitution of the machine for some works of the brain. The second wave provided artificial arms. The third provides artificial brains producing artificial intelligence. We are just in the first decennies of the third wave but it is important to underline some performances of the artificial intelligence. A computer built in order to solve some problems of euclidean plane geometry discovered a solution unknown from the mathematicians who built the machine. It was not an inhuman solution because afterwards it was noticed that this engine-solution had been published by a real man some centuries before, but was just not known by the builders of the computer, who were therefore not able to introduce the solution in the engine. The parallelism between the mathematician who published the solution some centuries ago and the computer that rediscovered it independantly is dramatic : trained to solve some problems of plane euclidian geometry, both were able to discover a solution not directly communicated to them, and the builders of the computer admit that this computer-solution was more elegant than any solution known by them. Is that not an extraordinary demystification and demythification of some mathematical reasoning ? The second wave is marked by the enormous industrial concentrations in which the workers play often the role of mere robots. The mathematics of the first wave are clearly the geometry of the candid idealized physical space. The mathematics of the industrial period are clearly that of calculus, matrices, vector spaces ... and finally all the great structures emphasized by Bourbaki seem to echo the industrial complexes. Let us remember that in the last fifties, the mathematical structures were compared to engine-tools by some of the promoters of the New Math.

The mathematics of the second wave broke the original euclidean jail. But the need of a well defined territory of spacial character appears in the regeometrisation of the so called modern mathematics,which is emphasized by the creation of all a collection of spaces : vector spaces, topological spaces, Hilbert spaces, unitary spaces, Banach spaces, Fréchet spaces, differential spaces, dependancial spaces ...

The comparison of the measurement of the time under the three waves is fascinating : the clepsydras, sand-glasses and sun-dials of the first wave are succeeded by the mechanical clock of the industrial wave and finally by the digital-clock of the third wave where the mechanism has been replaced by a quartz and a microprocessor, which consists in fact in a concret microdiagram. The informatic clock is autoscheme. It is reduced to its own conception. Let us call it *conceptual*. The example of the measurement of the time is general. The third wave replaced a lot of weighty mechanisms by light conceptual electronics. The gigantism of the second wave is replaced by the miniaturisation. The soul of the informatics is made of concrete arrow-diagrams.

The mathematics of the third wave is that of the most conceptual aspects of the great abstract structures (emphasized by Nicolas Bourbaki). They are essentially relational and sagittal. One synthesis is obtained by the theory of categories which is the mathematical theory of arrows, which are at the same time mathematical objects and teaching aids in mathematical education. As all the very important computational aspects of the mathematics of the second wave can be performed by the computers, the conceptual aspect becomes more and more important and fundamental. Because of the hand-calculators, it is not anymore important to teach a child to compute long numerical calculations, but the pupil has now to know more than before the meaning of the operations and of the other concepts of mathematics. Full areas of so called applied mathematics have become now quite obsolete.

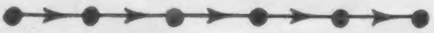
As all the computations and the languages of computers are exclusively games with the binary numbers (that is : numbers written in the binary numeration), those become a keystone of modern civilisation. And as they constitute at the same time a marvellous teaching aid to introduce to elementary arithmetics, to the real numbers and to the theorem of Thales, they are fundamental for both computational and conceptual aspects of mathematics. Any syllabus that would not emphasize their importance is definitely obsolete and underdeveloping.

The computer is not only an engine giving solutions of equations, it may be an artificial intelligence that can have a conversation with you. In that purpose it is important to speak to the computer in its own language, and necessary to understand this language and to understand how its artificial intelligence apprehends the different concepts. This is in fact a new way offered to mathematical education. Trying to understand the computer comprehension, helps the child to understand how himself he understands, which is what we shall call an epistemological knowledge of mathematics. In the past it was often difficult to justify the necessity of rigour in mathematics. In a dialog with a computer rigour becomes essential, because the computer understands only what is stated. Any deviation from a correct formulation, produces a deviation in the understanding, and as a consequence the reaction of the computer will not be the answer to the intended question, but well to the deviated one ... (which may arise interesting problems).

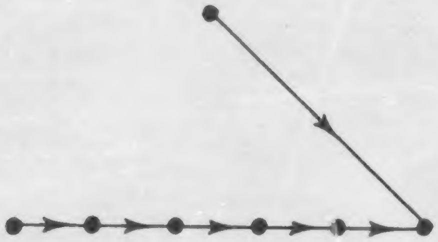
As they are so conceptual, mathematics are a dish full of psychology. The psychologist may help the educator to understand the process of understanding mathematics which is now important not only in mathematical education but in fact in mathematics themselves. Nevertheless let us begin by a reversed example where mathematics -

arrows - may help the psychologist and particularly the epistemologist.
Seriation means the ordering of a (finite) subset of a totally ordered set. As total order means a relation which is simultaneously *transitive* ($a \leq b \leq c$ implics $a \leq c$) *reflexive* ($a \leq a$), *antisymetric* (never $a \leq b \leq c$) and *total* (always : $a \leq b$ or $b \leq a$). The question arises of the role of those different properties in the process of seriation.
The answer of the problem is given by this theorem of *Redei* in which *totally related set* means a set equipped with a total relation T (all $a \neq b$ of the set verify $a\ T\ b$ *or* $b\ T\ a$).
THEOREM OF REDEI : *Any finite total relation contains (at least) one chain* $a_1\ T\ a_2\ T \ldots T\ a_n$ *where the* a_i *are different elements of the set.*
Describing children facing a seriation problem, some psychologists emphasized the fact that "*the pupils did not use transitivity*".
Those children were perfectly right because the theorem of *Redei* shows that transitivity plays certainly no essential role in the process of seriation, because it perfectly works even in the case of a non transitive total relation.
The theorem of *Redei* is a marvellous pedagogical situation and admits the amusing *Bus driver* version : In this island with a small number of towns, any two towns are joined by at least one one-way road (in one or in the other direction).
A bus driver asks if it is always possible to find an itinerary (for his bus) visiting once and only once each town and never coming back to the departure-town.
It is even more exciting, but in fact equivalent, (and only apparently more difficult) to suppose that between two towns there exists always one and only one one-way road (in one direction but not in the opposite one). In the latter case, the picture of the problem is just a complete polygon with one arrow on each side.
It is very interesting to let the pupils try to find a solution. It has been noticed that some problems of this type can be solved sometimes surprisingly quickly by handicapped people of 18 who were completely unable to write, to read, and could not recognize globally the number of elements of a set of more than four elements. This leads to conjecture that some problems with arrows use the less evoluated parts of the brain. Psychological research on this subject would be wellcome.
Let us now describe a proof of *Redei*'s theorem.
Take any arrow of the map. Permit us to draw it symbolically here

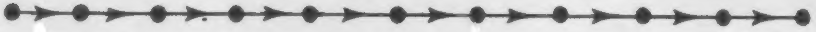
Expend it forward as far as possible in a road that never comes back to an already visited town, nor to the departure town. As the graph (or map !) is finite you will finally be blocked. Let us symbolically draw this maximal road as follows



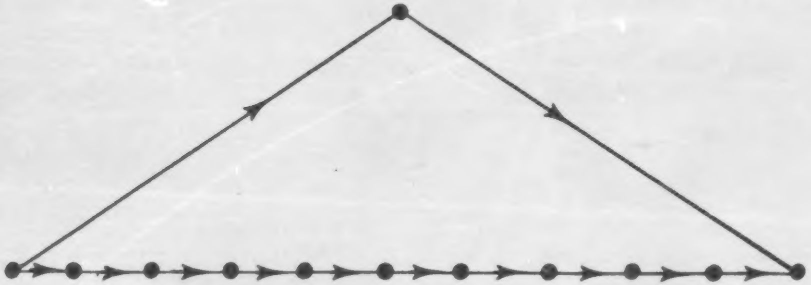
Clearly on the map, *in situ*, this road may have a much less straight aspect. But no matter : we use the implicit convention applied in the metro-cars, where the itinerary followed is often drawn straight. We know that the last point to the right, the town where we are blocked is joined by one arrow to all points (towns) of the graph; moreover for any town not yet visited by our maximal road, this arrow will be towards that extremity



We can do the same thing, dually, backwards, obtaining a bimaximal road

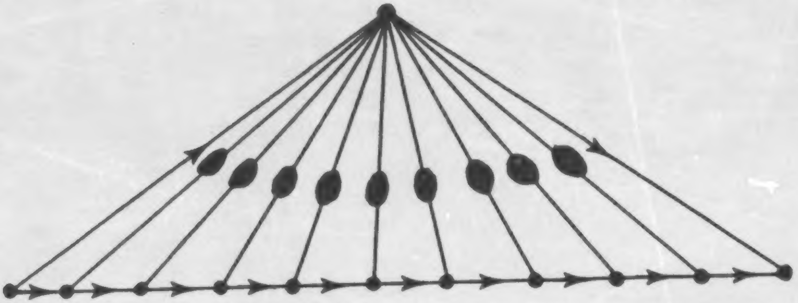


the bimaximality being translated



where the point above is any (if any) town not visited by the road (*visited* includes departure and arrival). If, by chance, our bimaximal already visits all the towns of the island, that's O.K., and we may go home. If not, then there is an extra town, not visited as pictured above.

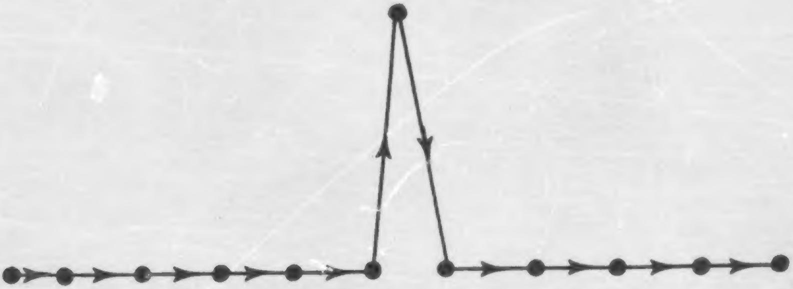
Moreover, we know that this extra town is joined to all the towns of our bimaximal road but we do not know the direction of the hidden arrows : up and down



But each of those "*vertical*" arrows is ↑ or ↓ , making from left to right, a finite sequence of ↑ and ↓ starting with a ↑ and ending with a ↓

↑ ? ? ? ? ? ? ? ? ? ? ↓

Consequently there must be in this sequence a ↑ immediately followed by a ↓ , ... permitting the detour via the extra town



The new road being again bimaximal, will permit a new detour by an extra town, if any. The towns being in finite number, the theorem is proved ■

This proof follows the procedure, sometimes described, of children facing the problem of the seriation of a finite number of Cuisenaire's rods of different lenghts. They build first a stair that they expand at both ends obtaining finally a stair going from the smallest rod to the longest one. Then they try to insert one by one the remaining rods.

In the road version of the theorem of *Redei*, towns and roads are concrete objects. The same situation arises in the abstract graph version where *points* and *arrows* are *objects* of the theory. On the contrary in the Cuisenaire's rods version the points (= rods) are concrete but the arrows are *virtual*. The opposite situation arises in the following arrowed dominos version, where the role of towns is played by the natural numbers *1,2,...,n* and the role of arrows by concrete arrowed dominos $\boxed{a \to b}$.

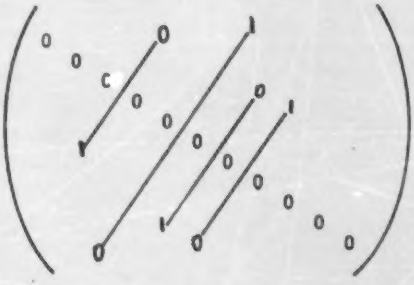
The arrowed dominos version of the theorem of *Redei* reads : *If a set* D *of such arrow dominos is such that for each pair* $a \neq b$ *of elements of* {[illegible],n} *one at least of the two dominos* $\boxed{a \to b}$, $\boxed{b \to a}$ *belongs to* D , *then* [illegible] *permits the construction of an arrow-dominos-route visiting once and only once each of the naturals* 1,2,...,n .

$\boxed{2 \to 4}$ $\boxed{4 \to 3}$ $\boxed{3 \to 1}$ $\boxed{1 \to 5}$ $\boxed{5 \to 2}$

The arrows or roads are the concrete dominos, but the towns or points are virtual: no *object* of the game represents a natural.

The theorem of *Redei* about a total relation defined in a (finite) set, has its *Cartesian* or *matricial* version :

If a finite square matrix consisting only in 0's *and* 1's *has a null diagonal and for non diagonal coordinates* $m_{ij} + m_{ji} = 1$ *then it is always possible by a permutation of the rows and the same permutation of the columns to obtain a matrix with the upper diagonal made of* 1's

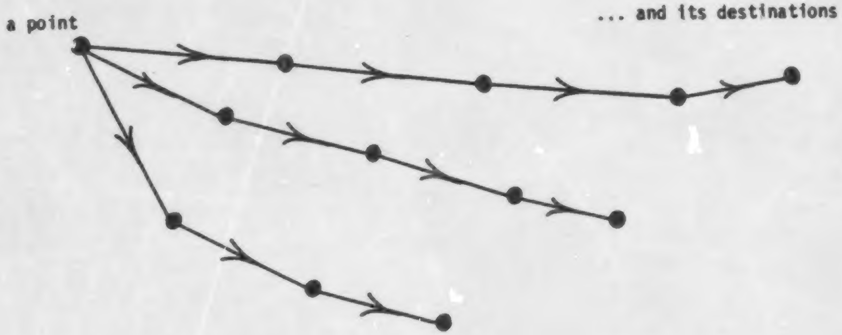


$$\begin{pmatrix} 0 & 1 & & & & & & & & & \\ & 0 & 1 & & & & & & & & \\ & & 0 & 1 & & & & & & & \\ & & & 0 & 1 & & & & & & \\ & & & & 0 & 1 & & & & & \\ & & & & & 0 & 1 & & & & \\ & & & & & & 0 & 1 & & & \\ & & & & & & & 0 & 1 & & \\ & & & & & & & & 0 & 1 & \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \end{pmatrix}$$

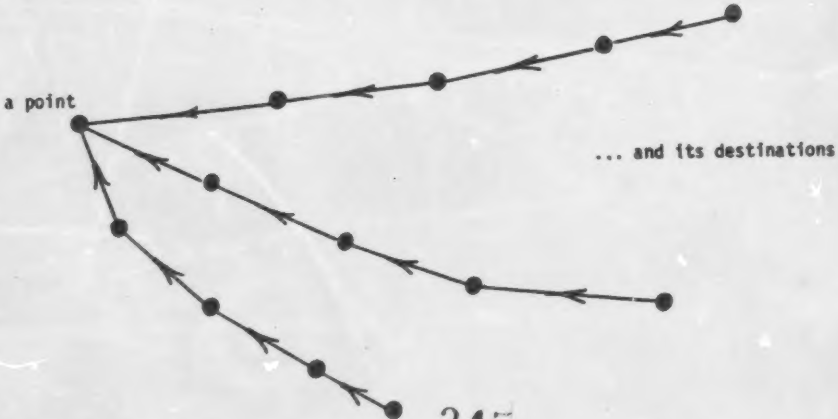
The road- , graph- , domino- , matrix- version of the theorem of *Redei* are isomorphic and equivalent. The rod- version is an order-relation particular case, but the process of seriation is essentially the same in all of these versions.

As *seriation* has been considered so important by many psychologists and namely by by *Piaget*, it seems reasonable to test the reactions of children, adults, - handicapped and not - in front of the different versions of the theorem of *Redei* and its proof. As you may guess my conjecture is that the graph (or road) version is the deepest one; that is : accessible by more children and even by the slowest ones. The importance of arrows - not often explicity used by *Piaget* appears in what could be the *Piaget's* groupings.

In any arrow diagram let us call *destinations* of a point, all the points reached from the given point by arrow-routes of the graph

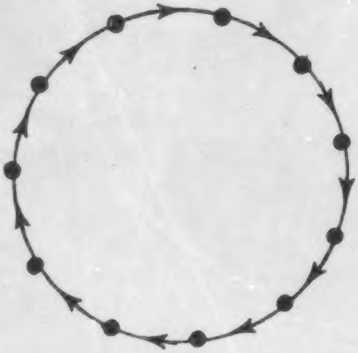


Antidestinations are defined dually:

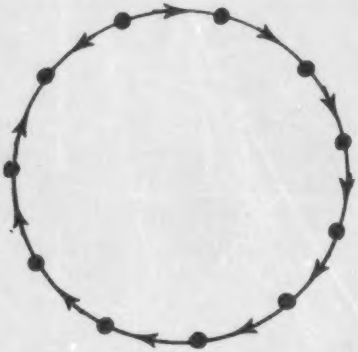


Piaget's groupings are the finite arrow diagrams

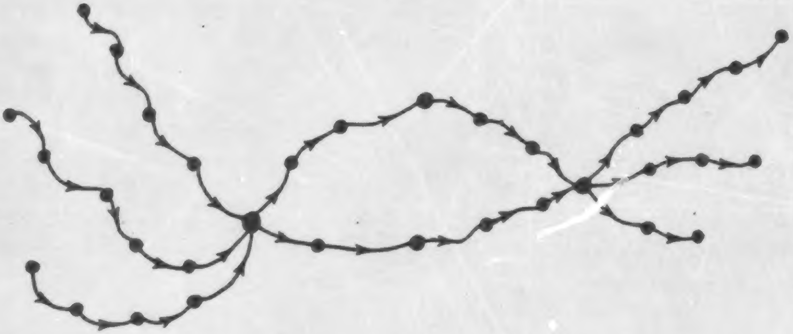
without *cycles*



without quasi cycles



in which any pair of points have a *nearest destination*, (the other destinations of the pair being the destinations of the nearest destination of the pair) ; and dually.



(different dots printed for destinations or antidestinations may represent a same destination or antidestination.)

In the frame of lattice and dependance theory, the groupings can be redefined : *finite lattice bases* (that is : minimal generating sets of a finite lattice or, equivalently, free generating sets of a finite lattice).

These two definitions of the groupings are interesting, and also their comparison. *Piaget* claimed that the great mathematical structures correspond to preexistent mental structures, and he emphasized the importance of the groupings in the psychology of intelligence. The second definition of the groupings given above is verbal, esoteric, doesn't speak to the layman, but locates precisely the groupings in the *repertoire*, as in music, of nowadays conceptual mathematics. On the other hand the first arrow definition of the groupings can be communicated to the man in the street without any kind of previous knowledge in mathematics.

The importance of groupings underlines the importance of arrow diagrams. The verbal definition speaks to the left brain of a mathematically cultured man of to-day. The second permits the global understanding of the concept. It speaks directly to the right brain but also to the first and second brain of the man.

Mathematics are not a game for pure intellectuals and are not produced nor by the left brain alone, neither even by the whole cerebral cortex alone. Mathematics are a concert played by the whole brain, including the second mammal brain and the first reptile brain of the man. The latter is considered as responsible of the territory concept, hole digging, sexual impulses, rites, hierarchies ... all topics fundamental in the conceptual mathematics of to-day. The need and confort of a territory appears in the constant search of mathematicians for homes like all kinds of spaces, or structures : groups, categories, vector spaces, lattices ... groupings. Isn't clear that the phallic symbol constituted by the arrow arises from the deepest first brain ?

It is very convenient to picture all kinds of hierarchies, relations, categories, groupings by arrows. They are used in organigrams, sociograms, flow charts and constitute a great pedagogical aid. But they are at the same time genuine mathematical objects by themselves. They are deeply rooted not only in the human being but also in the artificial neurology of the computer. A mathematical education adapted to informatics cannot avoid an intensive use of arrows.

Chemical facts are independant of the man, while their relation is partly human and may be deeply influenced by the chemical human traditions. In mathematics the factual part is much less important and most of the mathematics are a human creation. This situation could be compared with that of music. Sounds are independant of the man, and so are the songs of the birds, or the music of the wind in the trees. Philosophers may speak of music without reference to any musician. But the music of to-day is inseparable of the history of music, and any musician has to study the *repertoire*, because music is a culturally cumulative activity. The same appears in mathematics, which are quite inseparable of the history of mathematics and at the same time of the social context. At any moment of their history, mathematics partly herited from the past, seek for new intuitive roots and homes in the social contemporary context, psychology and ideology included. It is in that optic that new researches are indispensable for the improvement of mathematical education. They must come altogether from all kinds of psychologists, epistemologists, informaticians and mathematicians. Mathematics are more and more not only a precious tool for the understanding of sciences and the mastery of technics but an indispensable help for a balanced work of the brain : a factor of human happiness.

A NEUROPSYCHODYNAMICAL THEORY OF MATHEMATICS LEARNING

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1. *Introduction; mental achievements as networks*

Recent findings in the Neurosciences throw some light on the working of the human brain as it stores and brings forth mathematical knowledge. We develop an epistemological-psychological picture of mathematics consistent with these findings, adding a number of (falsifiable) conjectures derived from mathematics as a *creative* and *discovering* activity.

Every conscious mental activity in animal and man, aimed at achieving a certain immediate or distant goal – as these findings support – integrates a number of, dynamically stored, former achievements into a new one, *viz.* the mental achievement corresponding to that goal. Thus, every achievement is itself a *network* linking other achievements. On this view, brain development, starting at the moment of conception, is a confluence of networks of different sorts. Some come into being as "primitives", *i.e.* they are preprogrammed as innate senses or instincts, but in man almost all other achievements are learned. There are those that, once learned, may be changed or copied, as is the case when we have stored many mental pictures of a triangle, much in the same way as we all use the number "2" many times over in different contexts of achievement.

Usually however, we copy achievements only partially, when integrating them into new ones, since we tend to "throw out" inessential concomitants of the former, making it unnecessary to store every individual achievement.

For instance, not only instinctual achievements but also learned ones are often stored on a long term basis as moulds or *templates for routine use*, so that it becomes quite unnecessary to store every individual cast of the template. Hence, *skills* of all sorts are achievements, that, even in pure mathematics, involve the senses, be it only because there pencils are indispensable. All mathematical discovery integrates former skills into new networks of reasoning. These networks may be discarded, changed and learned by others, eventually producing new skills. We ask : "*how* did you arrive at this or that result ?", so that we may be able to re-achieve the same, but in a different set of brains. In pure mathematics we find the strongest bifurcation anywhere between template and cast, in that the former is made part of a formal language, whereas the latter as an example or as a model of the theory leads a life relatively independent of the template.

It is noteworthy that the individual examples may be very rich in content, since they may involve a lot of concrete imagery drawn from experience. A "model" for a theory is a set-theoretically organised structure of examples. There is a neurophysiological basis for the duality template-cast in mathematics as we shall see.

There is also the question of "memory". Several kinds of memory are being discerned today : cognitive- and motor-memories, short- and long-term memories etc. Emerging new physiological insights into these kinds of memory are precious since they start to relate memory to neural networks as well as to biochemical processes. They are conducive to understanding the connections between the two different levels at which mental achievements and operations can be studied, *viz.* (i) the *micro-scale level* of (brain)physiology, and (ii) the *macro-scale level* of behaviour and functional experience, psychology, epistemology etc. including pedagogy.

In light of the present state of knowledge in brainphysiology relating to behavioural function, we may describe the central task of *neuropsychology* (and neuroepistemology) as follows :

> *Find those two-way mappings between the macro- and micro-level of the functioning of the human brain which map*
>
> mental achievements *to* neural networks
>
> *and*
>
> mental activities *to* flows in neural networks

It should be noted that we do not *identify* mental achievements with neural events, thus avoiding the trap of philosophical materialism.

We believe that progress in the neurosciences is often blocked by simplifying philosophical choices. We shall say that neural networks are the *substrate* of mental achievements, indicating thereby that neural networks exist, develop and extend hand in hand with mental networks; whence the mathematical term "mapping". These mappings are not to be construed as in some sense one-to-one, since one and the same mental achievement may be mappable to many different networks. For instance, after the necrosis of certain brain tissues, other tissue may take over the former's function. The enormous high degree of connectivity within the cerebral cortex grants it a flexibility and rerouting capability, gibing with any one-to-one speculation.

2. On the difference between "wiring" and "circuitry"

The word "network" refers of course to the "wiring" of the brain, *i.e.* to the manifold nerve fibres that connect the nerve cells or neurones. There are mainly three types of wiring in the brain, *viz.* (i) the *projection fibres*, including those connecting the older and lower centres with the "newer" part or *neo*cortex, (ii) the *commissural fibres*, which run mainly between "homologous" areas of the neocortex of the two cerebral hemispheres, and (iii) the *association fibres*, including the longer ones connecting distant areas in one and the same hemisphere as well as the short ones between adjacent areas (cf. B. Kolb and I.Q. Whishaw, 1980, ch. 20). The *wiring* of the brain, in the first instance complete at birth, undergoes, in the second instance, a further myelinization and a further connecting-up with neurones in later life. Through the wires, at any point during life, run the seemingly random chemo-electric pulses, that are controlled by the "pacemakers" of the thalamic nuclei in the midbrain. The well-known E.E.G. only picks up an averaged-out picture of these pulses, as they define electrical changes in the neocortex.

A *circuit* on the contrary, is a subnetwork of the wiring that has been "potentiated", *i.e.* for a shorter or longer time it has been made to support special, repeatable, non-random reverberating firing patterns, in such manner that it maps to a learned ability (or achievement). The change from mere wiring to circuitry is being studied intensively nowadays (cf. T.J. Teyler, 1978). The words "axonal growth" and "synaptic potentiation" refer to changes of this kind, but much has still to be discovered. It is generally believed though that well-defined circuits map to achievements, thus defining the usage of the (neo)cortex as it supports instinctual and habituational tasks, learned activities, skills and achievements. The dynamic integration of networks of circuitry is thus believed to form the substrate of all human action and understanding.

A non-random special pulse firing is not necessarily seen on the E.E.G. but the onset of it usually is (desynchronization). It need neither be the result of consciously willed behaviour, nor is it necessarily of such nature that it maps to voluntary behaviour. There are obviously mental achievements arrived at un- or subconsciously.

If no special firings take place then circuits are in a holding pattern from which they may be recalled; they are dynamically stored. Long term memory (l.t.m.) is thought of to be encoded in such circuits; when the circuit is being fired, then information encoded in it becomes mappable to the mental level. However, circuits or parts of them may become "depotentiated", so that long term memories become hard to get at. It is rather *circuitry* that is preserved on a long term basis than isolated bits of information.

The role of the frontal lobes in higher mental function should be stressed. These lobes have been of primary importance for the *egocentric spatial relations* and movement planning of the individual. More generally, the coordination and integration of body motion, the stability-in-time of emotional and social behaviour as well as long term planning turn out to be disruptable by lesions in frontal areas.

By contrast, the *allocentric spatial relations* and body tasks turn out to be disruptable primarily by lesions in the parietal and temporal lobes (for a survey picture of the lobes see Kolb and Whishaw, 1980, ch. 1). However, frontal lesions do not significantly affect l.t.m., once they have formed. Short time memories (s.t.m.) however, are affected by these lesions, notably in an asymmetric manner, as follows. For example, left frontal lesions may affect verbal recency – which involves wirings running between left frontal, parietal and temporal areas –, and right frontal damage affects non-verbal (pictorial and musical) recency.

Taking into account existing studies on cognitive learning and long term memory, we *conjecture* that all l.t.m. is based on what we call "long term ideation" (l.t.i.). L.t.i. is that particular function of the mind which – at its strongest during the initial stages of one's life – maps to (flows in) large trees of thalomo-cortical and cortico-cortical fibres, *initialising* them and turning them into trees of circuitry in the process.

> Initializations of this kind may take place via the same feed-forward and feedback (l.t.m. and s.t.m.) loops as described in Eccles (1980, p. 175 ff.) on the basis of known anatomical connections and physiological evidence (Kornhuber, Bliss, Lømo etc., loc cit.). They are known to involve the "emotional brains" or the limbic system whith a principal role accorded to the hippocampus and the amygdala.

The most important example of l.t.i. is the discovery of one's own personal gifts and talents. It is a selfrecognition process, often followed by more or less clear plans to develop the gifts. Thus, on this theory, the ongoing growth of an individual's l.t.m. during life, is the superstrate of the dynamical extension and connection of l.t.i. circuitries into denser and denser networks until all the available wiring and firing patterns are exhausted ...

As a corollary to the conjecture, s.t.m.'s are circuits not connected to l.t.i.'s. They may or may not be written into l.t.m. (e.g. in a manner as indicated by Eccles). On the other hand, l.t.m. of bites of information (names, facts, etc.) are to be viewed as accessibility of small subcircuits within greater l.t.i. networks; according as a subcircuit forms part of a greater number of other subcircuits with the property that they can be independently and/or randomly accessed, the rememberance is more facile and fast. The access need not run via the frontal lobes, as we have seen (hippocampectomy). Since pronounced mathematical talent shows itself often at a young age, future neuroepistemological research on mathematical thought should start with youngsters in the lower grades, so that their development can be followed over an extended period of time (although the study of historical coryphées of mathematics may be of help, cf. §5).

3. *Neuropsychology and the status of the number and space concepts*

Since mathematics is termed an "objective science" with fairly simple basic concepts, the neural mechanisms to which these concepts map must admit of being tracked down. Whereas among mathematicians there is no disagreement on the spontaneous *use* of number and space concepts, philosophical considerations still tend to pit some form of realism against some form of idealism. There is the well-known chicken-and-the-egg-question : do mathematical concepts or templates arise from (the experience of) some given order in the world that is independent of the knowing subject, or is the order that we experience a result of *apriori* mathematical concepts applied to the world ? Let us tackle this question with our dynamical theory of circuits. We look at natural number and space concepts.

Number has an *ordinal aspect* produced by such sequential neural firing patterns and processing modes, that map to sequences of spoken or written discrete signs or symbols etc. or to sequences of purely mental operations of the same (e.g. silent mental counting). Indeed, it has become known to what areas of the neocortex silent, digital counting maps, namely to the mesial part of the superior frontal cortex, just before the premotor area, in the *left* hemisphere (cf. Roland, Larsen, Lassen and Skinhøj , 1980, fig. 5 and 7). One cannot, however, say that *all* counting maps to the left brain, since for example *counting musical tones* maps dominantly to right hemisphere areas, even if some of their left homologues are also slightly excited (Ronald, Skinhøj, Lassen, 1981).

It is to be noted that counting, calculating and concomitant picturing is *not* as such a mathematical activity presupposing a number concept. But mental discernment of lawlike features involving *all* numbers (as, for example expressed in the Peano axioms), and reasoning with them, certainly *is* a mathematical activity.

Number has also a *cardinal aspect*, showing up when one forms mentally-creative collections or *sets* of things, leading to "magnitudes" and the cardinal of a set. This is a *holistic* processing mode, binding the things into a pictorial conceptual unity. If this binding together into a set takes place in extrapersonal space (usually involving the senses), then it is known that neural mechanisms in the *right* parietal areas are involved. If the act is purely mental, without language parallel, and with the ears plugged and the eyes closed, then the involvement of these areas is still conjectural (but extremely probable). However, there are indications that in that case there is a greater blood flow in a disctinct part of the right superior frontal cortex (Roland and Larsen, 1976).

Modifying slightly our former *complementarist* view (cf. Kuyk, 1977), we summarize as follows.

Since natural numbers are at once cardinal and ordinal numbers, there is only one irreducible concept of natural number, involving the whole brain, and mapping to the two principal complementary processing modes of the cerebral hemisphere.

Number Theory is that mental activity of the brain aimed at the discovery of lawlike features governing the natural numbers and at expressing or encoding them in a logical-linguistic framework.

> Comments. Our theory unifies creative as well as discovering features. In a very real sense, radical Formalism and Intuitionism tend to reduce mathematics to templates that map to circuitry in the left hemisphere ("idealism"). Cantorism claimed the reduction to imagery and sets ("realism"). Our number concept is falsifiable. For instance, on our theory, splitbrains, who before callosotomy did not have a number concept to begin with cannot be supposed to form one after the operation. They cannot do any creative mathematics; they may at the most repeat and extend the formal language part of their previous knowledge. They may be able to improve their counting and their picturing ability, but the connection between these is seriously disturbed. They may do the machine-like parts of mathematics rather well. Note that no computing- or thinking machine is in the offing of which one expects to form a number concept all by itself. Hilbert in his Grundlagen der Geometrie in a sense proposed the systematic development of the two processing modes of the hemispheres separately by keeping them in opposition, thereby according the right hemisphere an appositive role. He discovered topological concepts in the process.
> Further, Cantor's continuumhypothesis is an expression of the two different processing modes. Finally, Gödel's well-knwon theorems say, in effect, that formal language operations (i.e. abilities mapping to processing modes in the left hemisphere) are significantly poorer than the language of spontaneous number theory; a fact, we submit, that is due to deleting right processing modes from the spontaneous theory, which integrates the two from out of a higher level of operation.

The mathematical *space concepts* have also aspects mapping to the two cerebral hemispheres. Firstly, they all have a *pictorial aspect*, even if the dimension is high, and the space as a whole cannot be represented within the neural networks that are primarily reserved for sensorial-visual patterns and memories, in the right parietal-temporal lobes. It is known from splitbrains that for the formation and preservation of sensorial-visual and sensorial-auditory "pictures" the right hemisphere does not need the left one. On the other hand, it is not known to what extent splitbrains and normals use the same lobes for imaginative-creative imageries. At any rate, producing pictures or having them is not as such mathematics. There has to be a complementary *analytical-logical aspect* in order that one may speak of mathematics. That analytical aspect is supported by networks in the left hemisphere, which is the primary "seat" of logical reasoning and linguistics as well. Mental operations are manifest in the multiple ways in which the left brain acts on the right one, whereby sets of spatial pictures are being formed (points, low-dimensional figures, curves, shapes, spatial motions etc.) to become part of the highly personalized imagery of a mathematician, while giving holistic *meaning* to his abstract language. It is clear that in mathematics the left brain is "in charge", even if features abstracted from sensorial and imaginative pictures are often conducive to the upbuilding of a theory. As has been suggested before (Kuyk, 1977, ch. 3) : mathematical language and imaginative "models" mutually define each other in a dialectical process. The reader may add comments to this as we did after dicussing the number concept. It is important to note that spatial pictures in mathematics may range from Gestalt images (as in Classical Geometries) and loosely connected point sets to continuously deforming topological entities (as in differential equations and dynamic theory). Also the auditory images, making a mathematician to "hear" his theory should not be underestimated. Presumably, these images map to the whole rang of association areas between the (primarily) right frontal, parietal and temporal lobes. Recent stereognostic investigations (Roland and Larsen, 1976), and those on voluntary movements in extrapersonal space cited already, are consistent with the complementarist view here espoused. Future research in this area hopefully will be done with a view to finding areas to which very elemental intentional mathematical acts map.

4. *Catastrophe models and the dynamics of some brain functions*

"Flows in networks", mathematically-physically speaking, evokes the picture of reverberations, interference, feedforward and feedback mechanisms, excitation and inhibition, smooth transitions and catastrophic jumps, input-output systems, control parameters etc. In the case of the highly complicated network of the brain one may expect all these features to occur, even if the mind partakes in the control. We assume from the outset that the maps of §1 are like "diffeomorphisms", i.e. they map the smooth part of processes to smooth images, but they map discontinuous jumps to discontinuous jumps.

In mathematical learning jump features are prominent : the sudden recognition of a pattern in problem solving, but also the discovery that certain features fit into a comprehensive formal framework, are examples. In a former paper (Kuyk, 1981) we have noted that switches between language and imagery occur in all levels of mathematical thought ranging from those where the cognitive depth is relatively little (as in problem solving, exercises, etc.) to those where already an extensive imagery exists (as when a model for a whole theory is involved). These transitional jumps were attributed to symmetries in subnetworks of the wiring of the brain, which, in fact, facilitate switching.

Thus the idea arose to model the higher functions of the human neocortex by means of a cuspcatastrophe (fig. 1). We suppose the reader to be familiar with the general meaning of this picture, cf. Zeeman, 1977. The lower sheet of the surface M is the "language sheet"; any mental ability mapping primarily to the left hemisphere may be thought on it. The normal control parameter is "analysis" as opposed to "synthesis", which gives the direction of holistic modelling, on the upper sheet, which contains the primary functions mapping to the right hemisphere. The area of M above the cusp is the area with tense thought; here the concentration cursor may jump to the other sheet following a Formalization/Axiomatization catastrophe (the arrow F/A) or a Creative Imagination catastrophe (the arrow CI). On the lower sheet, the farther the cursor moves away from the cusp the more machine-like the mental operations become. Likewise, the more the cursor moves away from the cusp on the upper sheet, the less amenable to formalization the imageries become. By the way, once a theory has been established then jumping has come to an end and one can "walk" from language to model via "normal". What once took place at a great "cognitive depth" for someone has now become "trivial".

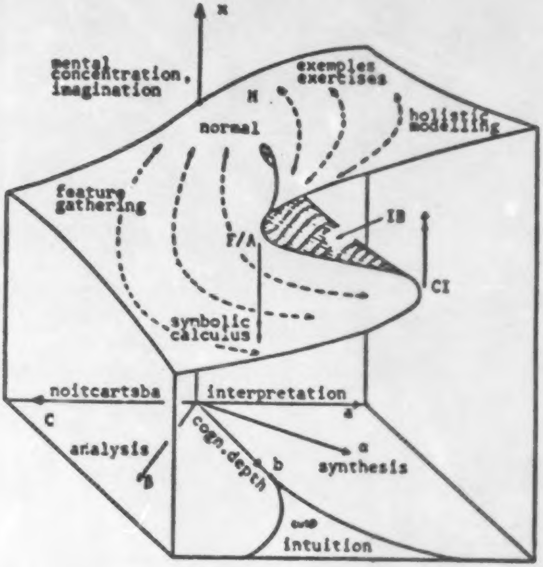
x
mental concentration, imagination
exemples exercises
M
holistic modelling
normal
feature gathering
IB
F/A
CI
symbolic calculus
noitcartsba
interpretation
a
C
analysis
cogn.depth
α
b
synthesis
β
intuition

Fig. 1 : Cuspmodel of mathematical concentration and discovery. It does not model the motivational drives that make the thought cursor move on the convoluted surface M, but not on the Inaccessible Behaviour part.

We verify the five characteristic features of a cuspcatastrophe, *viz.* sudden jumps, divergence, inaccessibility, hysteresis and bimodality (Zeeman, 1977, p. 18). Sudden jumps are part and parcel of discovery (cf. Hadamard, 1945, passim).

Divergence takes many forms. A very interesting one is "proof by handwaiving" as diverging from "formal proof". Handwaiving is an holistic language (body language) which neuroscientists know the right cerebral hemisphere is well-equipped to produce and understand. A handwaiving, muscleflexing researcher proves that he is working from, and acting on, an egocentric model. His task is to translate the features carried by the handwaiving into a more formal, and hence better communicable language, so that he can be checked by others. Every mathematics teacher should use, at the right time and place, handwaiving, to communicate images etc. since purely formal proof is only part of the whole mathematical story.

Inaccessibility refers to the IB-part of the sheet M, where the cursor cannot run. An example is the gap between the discrete integers and the geometrical continuum. Let us cite Fraenkel (1958, p. 198) for this :

> To understand the nature of the problem one should stress the fundamental difference between the discrete, qualitative, individual nature of number in the "combinatorial" domain of counting (arithmetic) and the continuous, quantitative, homogeneous nature of the points of space in the "analytical" domain of measuring (geometry). Bridging the abyss between these two heterogeneous domains is not only the central, but also the oldest problem in foundations of mathematics and in the related philosophical fields (two underlinings deleted by the author).

Hysteresis is the often long delay between the first glimmers of insight into how one can view things in terms of a processing mode of the other sheet, and the total break-through making the cursor do the final jump to the other sheet.

Bimodality obtains everywhere where a formula or statement admits of a direct interpretation in terms of imagined or real models. Every theory has also language elements that do *not* admit of direct interpretations. The loose connection between theory and model has often led to the discovery of new models for an existing formal theory.

5. *Odds and ends*

In light of a "dynamical theory in circuitry" the usual textbook terminologies stating that mental functions are "lateralized" or "localized" or "more left than right lateralized" etc. are wrong. What is meant is that a certain fixed area of the surface of the neocortex is physiologically "active" when that function obtains. Or that *some* area (in the left or right cortex) is active, as in tachistoscopical and dichotic listening tests. Our network theory implies that the cerebrum remains *one* even after complete callosotomy (without the bisection of the massa intermedia). One simply is left with a mutilated network. However, in view of our infinitesimal knowledge of the cerebral network, the textbook terminology may be used with these provisions in mind. As a matter of fact, *any* mental ability maps to the two hemispheres, be it since the mind may act directly on the "liaison areas" of the neocortex (Eccles, 1982) or just because any excitation of an area *ipso facto* excites the homologous area (by way of the callosum), thus calling on the (or an-)other processing mode of the same ability.

> Sherrington's and Eccles' theory on the liaison brain, in view of the recent results by Roland, Larsen, Lassen and Skinhøj (1980), raises the extremely important question, whether the same experiment by the latter four authors, but applied to splitbrains, would also show increased blood flow in both the supplementary motor areas. Their theory implies that, in splitbrains, not only "cross cueing" may supposedly take place along the limbic areas, but also via the "mind". There is no conclusive evidence for this as yet. It seems that the mind is "helpless" without the brain.

The question is whether the latter excitation is strong enough to map back in a way so as to *enrich* the mental function. In any case, there seems to be a triangular pattern of interaction between the mind and the two hemispheres, whereby the mind usually leads the action by helping to *unfold* the potentialities of the brain.

In this light it is understandable that, *with respect to a given ability* people often show a so-called *preferred mode*. Such a preferred mode may have been brought about by educational (environmental) influences, or it may be a natural disposition and innate. (One should be very careful in calling it the one or the other in any given case). Poincaré's well-known *psychological typology* (Poincaré, 1913, 1914) of (the ability of) mathematics students ("logical", orderly, "types" *vs.* "geometrical", intuitive "types") really cuts ice against the background of the cuspmodel (Kuyk, 1981, p. 370). He had a keen understanding of the two main mathematics processing modes, although his use of the word "intuition" still smells very Kantian. There are mainly *two* kinds of intuition in mathematics, since one can break out of the cusp into two directions, making different kinds of progress in the process. In view of the many different tasks that map to any one hemisphere, the two corresponding preferred modes appear in many guises, such as : logical

vs. geometrical (Poincaré), algebraic *vs.* topological (Kuyk, 1977), discrete *vs.* continuous (*loc. cit.*) analytical *vs.* holistic, digital *vs.* analogue, verbal *vs.* non-verbal, quantitative *vs.* qualitative (Fraenkel, *loc. cit.*), syntactical *vs.* semantical, formal system *vs.* interpretation, pure *vs.* applied, template *vs.* cast, analytic mathematics *vs.* non-analytic mathematics (Davis and Anderson, 1979). A call to a return to non-analytic mathematics usually involves a call to return to applied aspects and the senses (Davis and Anderson, *loc. cit.*, Goodman (1979) and Hersh, (1979)).

Grattan-Guinness (1980), in a study of the *coryphées* of mathematical physics in the first three decades of the 19th century, discerns *three* preferred modes of thinking, *viz.* the algebraic one (e.g. Poisson, Laplace, Ampère, Carnot, Lagrange, de Prony), the geometric one (e.g. Euler, Fresnel, Fourier, Hachette, Monge, Lamé, Navier), and the analytical one (e.g. Cauchy, Duhamel, Liouville, Sturm). The list shows that "preferred modes" have nothing to do with the choice of discipline of mathematicians. One may be an algebrist with a geometrical way of thinking etc. On Grattan-Guiness' criteria (the use of pictures, the way these mathematicians prefer to introduce the integral, etc.), Gauss did not fit into any category.

We suggest that his analytic category, which is the smallest, contains the "bimodals". They show no clear preferred mode, since the projection of their "thought cursor" in the control plane C, just as easily breaks out of the cusp to either side. Presumably, they have their abilities not "lateralized" in the fashion of the 96% of all right-handed people (on whom the design of fig. 1 is based). This raises a lot of questions.

For instance, psychological investigations on handedness and sex (Springer/Deutsch, 1981 and Craig, 1980) suggest that the remaining 4% of right-handed people and non-familial lefthanders (and perhaps most women) might be *bimodals*. There are indications that these people have more language ability mapped to the right hemisphere and more spatial ability to the left. (It is an unresolved problem whether greater lateralization means also greater ability). As an example, a bimodal, when presented with a mathematical function, might be better able to plumb (and synthesize) simultaneously the nature of that function as an algebraic entity, and as a graph (in space, for example), than a unimodal. So Grattan-Guinness' "analytic category" should contain relatively many from the remaining category of 4% righthanders. It is interesting anyway to know which mathematicians mentioned in this paragraph are natural lefthanders. As a kind of hypothesis we propose that bimodals have not one but *two cursors* running on M, and hence have an ability of *parallel processing*. This hypothesis is supported by investigations on left-handed people, who were confronted with (dichotic) rhythm tasks. Out came the surprising result that left-handed subjects differed significantly from the performance of right-handed subjects in a task that involved the analysis, synthesis, and response to dichotic rhythm patterns.1) "Possibly, the left-handed subjects are better able to analyze simultaneous stimuli and integrate differential information from them into a coherent structure. A bi-dominant theory of left-handedness would support such an interpretation of the data" (Craig, 1980, p. 618).2)

The relation between Grattan-Guinness' two other categories and the cuspmodel is clear. Let us add to his groupings, partly as a kind of speculation, some other names. To the "algebraic category" belong : de Fermat, Newton, Poncelet, Kummer, Weierstrass, Möbius, Kronecker, Dedekind, G.D. Birkhoff, Polya. To the "geometric category" : Descartes, Leibniz, Dirichlet, Galois, Steiner, Klein, Cantor, Hamilton, Hermite, Hadamard, E. Noether, Weyl, Einstein, Brouwer, Thom. Further bimodals could be : Gauss, Jacobi, Riemann, Poincaré, Hilbert, Weil. Until we have no clearer scientific ways of discerning the exceptional right-handed people and the non-familial left-handed subjects, and know more on their peculiarities, is all our categorizing a blow in the air, just as most present-day knowledge on the relation between sex and mathematical ability is.

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1) Right-handed people have a right brain dominance for music appreciation and a left brain dominance for dichotic rhythm tasks (cf. Bever and Chiarello, 1974). Professional right-handed musicians show left brain dominance for music, however, probably, since they listen "analytically" to music more than non-professionals.

2) A cursor may be regarded as the "center of gravity" of an "active" subnetwork of the brain, to which a conscious mental activity maps at any given moment. Thus, the dichotic tests suggest, that for some time at least, a task is being done in each hemisphere independently, until they join, leading to one cursor again.

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III. SHORT COMMUNICATIONS - VOLUME II

- A. CONCEPT FORMATION -

COGNITIVE PSYCHOLOGY AND DIDACTICS :
SIGNIFIED/SIGNIFIER AND PROBLEMS OF REFERENCE

Gérard VERGNAUD

C.N.R.S.

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Many cognitive psychologists involved in research on mathematics education do not refer to conceptions and concepts in mathematics, but rather to other theoretical frameworks such as logical structures, information processing, psycholinguistics and others. My view is that research in mathematics education cannot be fruitful if it does not pay enough attention to mathematical concepts. On the other hand, many researchers in maths education do not devote enough interest in the interaction of students with situations, and in the important duality signified/signifier in the use of mathematical symbols. Last but not least, the fundamental part of invariants in the growth of knowledge is not sufficiently recognized.

This paper aims at clarifying such problems. For an interactive conception of concept formation, a concept can be considered as a triplet.

$$C = (S, I, \mathscr{S})$$

S = the set of situations that make the concept meaningful
I = the set of invariants that constitute the concept and insure its stability under certain classes of transformations and variations
$\mathscr{S}$ = the set of symbolic representations that may represent the concept and the situations in which it is involved.

For instance the concepts of number, addition and subtraction refer to a

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large set of situations and classes of problems ; this set is only partially referred to in ordinary teaching, and has not even been completely studied by researchers. Different invariants can be distinguished (i.e. different relationships and different properties of these relationships), the comparative complexity of which can be studied over a long period of students' development. Different systems can be used to represent (symbolize) these relationships, depending upon the situations and their characteristics.

Another example is the concept of rational number, whose meaning outcomes from multiplicative relationships, among which inclusive fractions and exclusive fractions, scalars and functions, ratios and rational quantities play different parts and must be synthetised into the concept of rational number.

Other examples concern the concepts of function and variable which are very often prototyped by the linear function model, whereas other models and situations could be fruitfully used.

I will take the example of additive structures and try to analyse different kinds of misunderstandings that may contribute to dig the gap between teaching and learning.

First, the initial conception of children regarding addition and subtraction is shaped by situations which have nothing to do with the binary law of composition of natural numbers. The very first situations that make addition and subtraction meaningful to children involve the increase or the decrease of a certain quantity (sweets, marbles, chips...) and are adequately modeled by a state-transformation-state relationship, in which the set of directed whole numbers operates upon the set of natural numbers.

Second, quantities are not numbers but quantities in the same way as physical magnitudes are net real numbers but magnitudes, with their dimensional aspects. It would not be meaningful to add the number of girls and the number of boys if boys and girls could not be considered as subsets of the set of children. This dimensional problem is still more important for multiplicative situations, but "addition" and "subtraction" of "symbols" written on a sheet of paper may be meaningless to children, if they cannot figure out what sort of "quantities" are involved.

Third, there is a big difference between putting a problem into equation and writing the problem solution. Most children can write the sequence of operations they have made to solve a problem. This writing does not usually simulate the problem nor a numerical relationship, but only their procedure.

Let us take examples :
"Peter has just spent 15 francs to buy cakes, he has now 12 francs. How much did he have before ?"
Suppose a third-year student writes the answer

$$12 + 15 = 27$$

Does this writing mean that the student thinks that the composition of 12 and 15 is the same thing as 27 ? Not necessarily and we should require two more criteria :

a. 15 + 12 should be accepted is equivalent to 12 + 15
b. the equality sign should be a symmetric relationship.

Why ?
Suppose the meaning for the child is :
I start from 12, I add 15, it outcomes 27. None of criteria a and b is satisfied.

This procedure cannot be interpreted in the binary model, because it is an external operation of + 15 upon 12. And as 27 is the outcome, it would be meaningless to write 27 = 12 + 15

Moreover the writing 12 + 15 = 27 or even $12 \xrightarrow{+15} 27$ do not simulate the problem but only the procedure to solve it.

Problem	Procedure
$\square - 15 = 12$	$15 + 12 = 27$
$\square \xrightarrow{-15} 12$	$12 \xrightarrow{+15} 27$

The last schema shows that the procedure consists in inverting the direct transformation - 15 into + 15 and applying it to 12.

Let us take another example

Betty had 16 marbles before playing. She plays two games. At the second game she wins 4 marbles.
When she counts her marbles in the end, she finds only 13 marbles. What happenned during the first game ?

This is a difficult problem for children but it can be solved by some fourth and fifth-grade children.

Suppose a student writes

$$13 - 4 = 9$$
$$16 - 9 = 7$$

Betty had lost 7 marbles.

One can easily see that the first minus sign means the inversion of the direct transformation + 4 whereas the second minus sign means a difference between the initial state 16 and the intermediary state 9. This is another kind of ambiguity.

If another student, at the beginning of secondary school for instance, writes

$$16 - 13 = 3$$
$$3 + 4 = 7$$

then the plus sign means a difference between two directed numbers of different signs

$$x + (+ 4) = (- 3)$$
$$x = (- 3) - (+ 4) = - (3 + 4)$$

And of course these problems will be still more complicated in the study of the real number line and the famous equations

$$\overline{AB} = \text{abs } (B) - \text{abs } (A)$$

$$\overline{AC} = \overline{AB} + \overline{BC}$$

My view is that equalities and equations do not fit equally well all situations met and handled by students, but only a few of them.

Different authors have worked at classifying addition and subtraction relationships. I will not recall mine here, but I would like to formulate the following requirement for teaching.

If addition and subtraction take their roots in a large variety of situations involving different concepts such as cardinal and measure, transformation, comparison and inversion, difference and complement, and combinations of these concepts, if they can be modeled by internal operations in $\mathbb{N}$, in $\mathbb{D}$ and in $\mathbb{Q}+$, but also by external operations of $\mathbb{Z}$ upon $\mathbb{N}$, of $\mathbb{D}$ upon $\mathbb{D}+$... and by internal operations in $\mathbb{Z}$ or in $\mathbb{D}$, then one should study very carefully all cases of possible misunderstanding.

The problem for children is to recognize invariants and use them in problem-solving. It is not equivalent for them to recognize and use :

. the direct transformation inference knowing the initial state,

  if I win marbles I add
  if I lose marbles I subtract

. the difference inference

  knowing the initial and the final state, the transformation is the difference $F - I$ if $F > I$ and then I have won
  $I - F$ if $I > F$ and then I have lost

. the inverse transformation inference

  knowing the final state, and the direct transformation, I inverse the transformation and apply it to the final state ; this gives the initial state.

. the complement inference

  knowing the number of girls a and the number of children b, the number of boys is what I should add to a to find b. It is also the difference b - a.

. etc...

Equalities and equations, Euler-Venn diagrams, arrow diagrams, the real number line do not simulate equally well these entities and these inferences, and I have come to the conclusion that we should use a variety of these systems, especially when students are introduced to a new relationship and a new class of problems.

Situations are *referents*, concepts, preconcepts and inferences are elements of the *signified*-level of representation, symbols of all kinds and symbolic manipulations are elements of the *signifier*-level of representation.

History of sciences teaches us two important things :

1 - construction of invariants, extension of them to new situations and reduction of their field of validity is essential.

2 - symbolic systems are big helps in handling mathematical, physical and other scientific objects, especially in problem-solving.

How can we transpose these considerations into mathematics education ? The first problem is to identify the variety of situations that make a concept meaningful and recognize the various operations of thought that are necessary to handle and master these situations.

It may be useful or even necessary to use different theoretical models and different symbolic systems to represent these situations and these operations, especially when they are introduced. It is also an important challenge for teachers to help students in making the synthesis of these different meanings and build the mathematical concept and structure that will cover all cases. For instance ($\mathbb{Z}$, +) and ($\mathbb{D}$, +) and the real number line for additive structures.

These different steps, and especially the synthesis, cannot be made easily. History teaches us also that big and long-lasting epistemological obstacles must be overtaken. Until the 19th century, even first-class mathematicians had difficulties in figuring out negative numbers and operations upon negative numbers.

My view is that it is more fruitful to look at these difficulties from an epistemological point of view, (i.e. considerations upon mathematical knowledge) than from the point of view of the logical structure of tasks as Piaget has done, or from the point of view of information-processing.

I do not mean that information-processing cannot enlighten some problems and especially limitations in the span of data and relationships that can be taken into account at the same moment, but it does not say enough about concepts and conceptions. It cannot explain, for instance, why the primitive model of addition and subtraction may raise difficulties when the problem comes of extending addition and subtraction in $\mathbb{N}$ to addition and subtraction in $\mathbb{Z}$.

Other examples, in multiplicative structures, will be given orally.

PROCEDURES DE RESOLUTION D'UN PROBLEME A 2 INCONNUES

Procedures for solving a problem with two unknown factors

M.N. AUDIGIER, J. DOUAIRE, J.C. GUILLAUME, J.L. PORCHERON

We purpose researching the way in which a procedure for solving a problem is worked out in a situation where the appropriate mathematical model is not available to the subject, and where he cannot refere to a known method. This paper presents the method used for analysing the conventions obtained in standardized individual entries and gives the first results concerning the 300 pupils of the fifth year of elementary school who are confronted with one or two problems with two unknown factors.

Nous nous proposons d'étudier la manière dont les enfants de la 5ème année de l'école primaire élaborent une procédure de résolution d'un problème pour lequel ils ne peuvent se référer à une méthode acquise et identifiable à partir de l'énoncé.
Le problème choisi repose sur un système de deux équations à deux inconnues :

> "Dans ma tirelire j'ai 56 pièces de monnaie.
> Il n'y a que des pièces de 2F et des pièces de 5F.
> Avec ces 56 pièces, j'ai 163F."

Cet énoncé a été proposé à près de 300 élèves. La moitié d'entre eux a résolu directement ce problème(G.N.), l'autre moitié après avoir cherché un problème de présentation identique mais dont les données étaient: 11 pièces et 31F(Problème P.N.).

Le recueil des données est fait par la méthode de l'entretien individuel (30 à 40 minutes par enfant) pendant lequel l'expérimentateur intervient de façon codifiée. Cette codification porte sur le moment de l'intervention(erreur de calcul, blocage de l'enfant pendant plus de 3mn, arrêt sur un résultat ne satisfaisant pas à toutes les contraintes de l'énoncé) ainsi que sur sa nature (rectification du résultat d'une opération, demande de relecture de l'énoncé, proposition d'une configuration quelconque, proposition d'une configuration de même nature mais différente de celle trouvée par l'élève). Cette standardisation des conditions de l'observation doit permettre des comparaisons plus fiables d'un protocole à l'autre.

METHODE D'ANALYSE

Le premier aspect de la recherche a consisté à mettre au point une méthode d'analyse des protocoles individuels. Ceux-ci ont été divisés en "séquences" après définition d'un certain nombre d'indices qui marquent des ruptures dans le comportement et qui sont à même d'indiquer une réorientation du processus de recherche. Ces indices doivent être aussi objectifs que possible et ne pas préjuger de l'interprétation du procédé de résolution.

RESULTATS GENERAUX

- 150 enfants ont fait deux problèmes: énoncé avec "11 pièces et 31F"(problème P.N.) suivi de l'énoncé avec "56 pièces et 163F" (problème G.N.).
- 143 enfants ont résolu directement le problème G.N.

Le problème P.N. est réussi par 65% des enfants. Le problème G.N. par seulement 26% et ce taux ne diffère pas sensiblement qu'il ait été précédé ou non par la recherche du problème P.N.
La taille des nombres a ainsi considérablement complexifié la tâche de résolution et le passage par le problème P.N. n'est pas globalement une aide pour la réussite du problème G.N.

PROCEDURES DE REUSSITE DANS LE PROBLEME G.N.

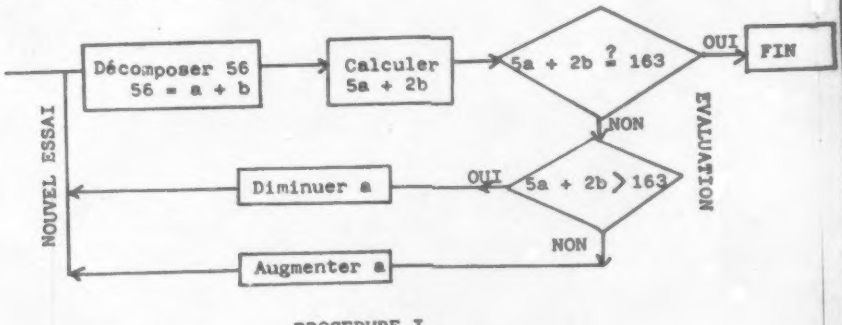
Sur l'ensemble des observations les procédures suivantes ont été relevées:
PROCEDURE I- 50% des élèves qui réussissent utilisent une procédure "à nombre de pièces constant". Ils font des essais de couples de nombres (a,b) qui vérifient tous la contrainte a+b = 56 jusqu'à ce qu'ils obtiennent la seconde c'est-à-dire 5a + 2b = 163

PROCEDURE II- 20% des élèves utilisent une procédure "à somme d'argent constante". Ils cherchent des décompositions de 163 sous la forme 163 = 5a + 2b jusqu'à ce qu'ils trouvent celle qui réalise a + b = 56 . Généralement la procédure consiste à choisir d'abord un multiple de 5 puis à déduire le nombre de pièces de 2F en calculant b = (163 - 5a) / 2

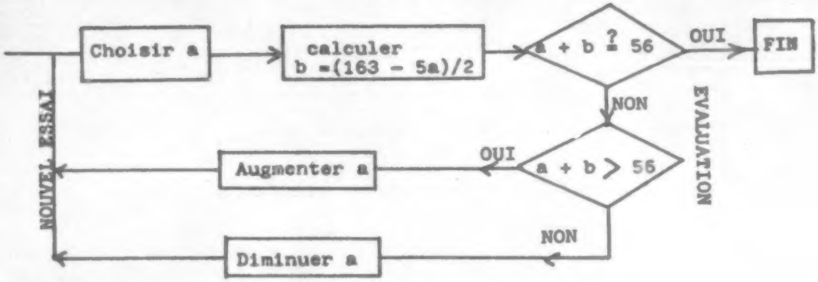
PROCEDURE III- 20% des élèves cherchent à réaliser en même temps S = 163F et P = 56, au cours d'approches ou d'essais successifs mais les couples (a,b) utilisés ne réalisent S et P qu'au terme de la résolution.

PROCEDURE IV- 10% des élèves donnent le résultat exact sans trace écrite de la méthode utilisée pour y parvenir.

Ainsi 70% des enfants qui réussissent ont une procédure qui limite le champs d'exploration des couples (a,b) ce qui simplifie l'évaluation de l'écart au but ainsi que la production d'un nouvel essai "meilleur". Ces deux procédures de résolution peuvent se représenter schématiquement:



PROCEDURE I



PROCEDURE II

Pour ces deux procédures le point crucial est le choix de la nouvelle valeur de a après l'évaluation de l'écart au but. Par exemple dans la procédure II il n'est pas évident de devoir augmenter le nombre de pièces de 5F lorsque le nombre total des pièces dépasse 56. Dans la procédure III la régulation des essais est encore plus difficile vu la diversité des configurations auxquelles on peut aboutir:

$a+b < 56$ et $S < 163$, $a+b < 56$ et $S > 163$, $a+b > 56$ et $S < 163$ ou encore $a+b > 56$ et $S > 163$.

Un indicateur de la difficulté à gérer une procédure nous est donné par le nombre moyen d'essais nécessaire pour aboutir:

	PROC. III	PROC. II	PROC. I
GR. 1 (P.N. et G.N.)	4.4(11)	4.2(5)	4.6(20)
GR. 2 (G.N.)	3.2(5)	3.8(10)	6.3(18)
TOTAL	4 (16)	3.9(15)	5.4(38)

Globalement les procédures II et III nécessitent moins d'essais que la procédure I. Dans ce dernier cas les essais sont assez nombreux pour le groupe 2: l'hypothèse que l'on peut faire c'est que le coût d'un essai est relativement faible et permet à certains élèves de réussir même si la régulation de leurs essais n'est pas parfaite.

ANALYSES DES ECHECS DANS LA RESOLUTION DU PROBLEME G.N.

Les procédures décrites plus haut ne mènent pas tous les enfants qui les utilisent à la réussite et il nous faut maintenant examiner quelles difficultés spécifiques ils ont rencontré.

CAS 1-L'élève utilise une procédure qui a mené d'autres au succès mais il l'abandonne après plusieurs essais: les essais ne convergent pas.

CAS 2-L'élève amorce simplement la procédure: pas de réitération.

CAS 3-Après avoir atteint un but (S ou P), la prise en compte de l'autre entraine la perte du premier.

	CAS 1	CAS 2	CAS 3
GR.1 (P.N./G.N.)	27	14	4
GR. 2 (G.N.)	11	10	3
TOTAL	38	24	7

Les cas 1 et 2 révèlent les principales difficultés:

- celle de la gestion des essais successifs qui peut devenir si lourde qu'elle entraine l'abandon de la procédure;
- celle du statut d'une stratégie par essai-erreur chez l'élève qui semble condamner une méthode qui échoue au premier essai.

Les difficultés signalées varient selon les procédures:

	CAS 1	CAS 2	CAS 3
PROCEDURE I	9	19	7
PROCEDURE II	8	3	
PROCEDURE III	21	2	0

Dans la procédure III c'est l'enchainement des essais qui pose problème; cette difficulté est aussi présente dans les autres procédures. Dans la procédure "à nombre de pièces constant" l'abandon de la méthode est le cas le plus fréquent: il semble qu' il s'agisse ici moins d'une esquisse de procédure que d'une tentative de proposer un couple (a,b) compatible avec une partie du problème.

Nous avons dit plus haut que les performances dans le problème G.N. sont globalement indépendantes du fait d'avoir ou non cherché le problème P.N. et que la réussite au problème P.N. est un faible indicateur de la réussite au problème G.N.. Pour compléter ces remarques regardons le taux d'accès aux procédures fortes (F) dans chaque groupe et intéressons nous au temps de résolution du problème P.N. comme indicateur de la réussite au problème G.N..

	Pas d'accès aux procédures F	Accès aux procédures F	Réussite
GR. 1 (P.N./G.N.)	43%	30%	27%
GR. 2 (G.N.)	59%	17%	25%

La résolution d'un problème numériquement plus simple mais de structure et de présentation identiques permet à davantage d'enfants d'accéder à des procédures intéressantes même s'ils ne sont pas toujours capables de les mener à terme.

Dans le tableau suivant on a les probabilités de réussir au problème G.N. en fonction du temps de résolution (réussite) du problème P.N. et l'évolution plus ou moins favorable selon la procédure utilisée: Très favorable = Taux de réussite >70%, Favorable= 40% < T < 50%, Plutot défavorable 30% < T < 35%, Défavorable 10% < T < 25%;

La première remarque que l'on peut faire est qu'une réussite très rapide (moins de 4 mn) dans le problème P.N. est une condition très favorable à un succes dans la résolution du problème G.N.. Au delà de 6 mn les chances de succès descendent très en dessous du taux moyen des réussites (T = 38%).

Nombre d'élèves	Réussite au Pb. P.N. en:	Taux de réussite au problème G.N.	Taux de réuss. dans l'utilisation des Procédures III	Taux de réus. dans l'utilisation des procédures I ou II
17	1 à 2 mn	71%	Très favorable	Très favorable
15	3 à 4 mn	53%	Favorable	Très favorable
25	5 à 6 mn	36%	Plutôt défavorable	Favorable
12	7 à 8 mn	17%	Défavorable	Défavorable
25	Plus de 8 mn	12.5%	Défavorable	Défavorable

On observe un net décalage dans le taux de réussite selon le type de procédure utilisée. Les procédures III déjà signalées comme difficiles à gérer restent moins longtemps favorables à leur "utilisateur". Ceci peut expliquer une partie des difficultés rencontrées par les élèves ayant utilisé cette procédure avec succès dans le problème P.N. et qui cherchent à la réutiliser.

ETUDE DE LA FILIATION DES SEQUENCES

Quelles sont les séquences précédant les séquences fortes (F') ? Peut-on déterminer certaines procédures à partir desquelles les transitions vers F sont facilitées ?

Parmi les 182 transitions aboutissant à une séquence de type F (concernant 144 enfants, soit 49% de l'effectif total) dont 76 mènent à la réussite, nous distingueront:

- les accès directs: la première séquence faite par l'enfant est de type F
- les transitions internee (F→F) : il s'agit de changements de procédures à l'intérieur des procédures de type F ou d'une reprise d'une séquence déjà faite après une séquence de transition.
- les accès externes: l'accès à F a lieu après une séquence de type différent.

Répartition des transitions/

	∅ → F	F → F	Extternes
Répartition globale	49%	19%	32%
% de transitions de réussite par type	50%	35%	34%
Répartition des transitions menant à la réussite	58%	16 %	26%

La moitié des transitions vers les procédures F sont directes; c'est dans ce cas qu'elles mènent le plus souvent à la réussite. Deplus si l'on ne prend plus pour référent l'ensemble des transitions mais les enfants nous notons quelorsqu'un enfant a cherché les deux problèmes dans 72% des cas (contre 52%) s'il atteint une procédure F il le fait dès la première séquence. Pour un élève, l'accès direct reste donc la modalité principale d'accès aux procédures fortes.

Les accès externes

Leur importance numérique est faible. Parmi les enfants n'atteignant pas dès la première séquence une séquence detype F seuls 29% d'entre eux sauront l'atteindre en cours de recherche. Peu d'élèves sont donc capables de faire évoluer suffisamment leur représentation du problème pour accéder à ce type de procédure dès lors qu'elle ne se présente pas immédiatement à leur esprit.

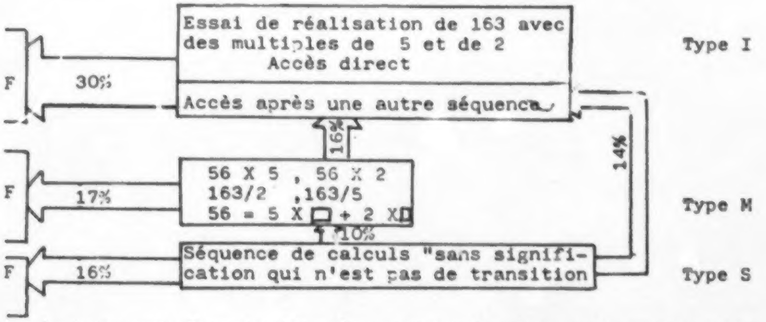
Répartition des transitions externes

UTILISATION DE DEUX TYPES DE PIECES		UN SEUL TYPE		Calculs "sans signification"sur les nombres ou des résultats de calculs antérieurs
Essai de réalisation de 163 par des multiples de 5 et de 2: 163 = 5X ☐ + 2X ☐	56=2x☐+5x☐	163 /5 163 /2	56X2 56X5	
49%	7%	15%	5%	24%

Type I ← → Type M → Type S

Près de la moitié des transitions externes se font à partir d'une séquence où l'enfant accorde la priorité à la réalisation de 163 à l'aide d'un modèle mathématique cohérent avec le problème; un bilan sur le nombre de pièces peut intervenir en fin de travail si 163 EST effectivement atteint.

Pour préciser les rôles respectifs de chacune de ces séquences dans le processus de recherche, nous allons les regarder dans leur ensemble :


Essai de réalisation de 163 avec des multiples de 5 et de 2
Accès direct
Accès après une autre séquence
Type I
F
30%
16%
14%
56 X 5 , 56 X 2
163/2 ,163/5
56 = 5 X ☐ + 2 X☐
F
17%
Type M
10%
Séquence de calculs "sans signification qui n'est pas de transition
F
16%
Type S

La séquence qui fournit l'accès majoritaire à F est aussi celle qui donne la meilleure chance d'accès. Toutes les autres séquences donnent un accès quasi uniforme nettement moins favorable. On peut penser que ces séquences constituent des séquences d'attente dont la nature propre ne contribue que peu à l'élaboration d'une stratégie de résolution. Elles permettent une entrée dans le problème: une prise de la dimension des nombres en jeu (56 x 5, 56 x 2), l'essai de l'instrument(division), le rodage du modèle des" multiples combinées" ou encore de meubler une certaine vacuité quant à la stratégie à adopter(calculs sans signification apparente).

Les séquences de calculs dits "sans signification" sont assez souvent utilisés. Parmi les élèves qui ne réussissent pas le pb. G.N. 62% l'utilisent au moins une fois s'ils n'ont pas cherché le problème P.N. et seulement 40% des autres enfants. Même certains enfants qui réussissent l'utilisent(13%).
Son utilisation majoritaire a lieu en début de recherche(74%).

Pour le reste elle constitue une séquence de repli suite à une impasse. Lorsqu'elle ne constitue pas une séquence d'arrêt spontané ou provoqué (15%), elle débouche très souvent (85%) sur une récupération de procédures l'ayant précédée ou sur une amélioration. Ceci permet de les considérer dans ce cas comme des séquences de transition. Notons que les séquences de division (163/2 OU 163/5) peuvent jouer dans le courant de la recherche un rôle analogue.

PROFILS GENERAUX D'EVOLUTION

Dans les tableaux qui suivent nous indiquons les transitions générales entre les premières séquences et les séquences les meilleures (séquences max) atteintes en cours de recherche.

MAX	62	28	22	31	143
Réussite	3	5	7	21	36
F	5	8	1	10	24
I	7	3	14		24
M	11	12			23
S	36				36
	S	M	I	F	ENTREE

63% 37%

Problème G.N. seul

MAX	43	13	33	61	150
Réussite	5	2	3	30	40
F	6	3	5	31	45
I	4	1	25		30
M	1	7			8
S	27				27
	S	M	I	F	ENTREE

37% 63%

Problème G.N. après P.N.

La familiarisation acquise lors de la résolution du problème P.N. donne donc un très net avantage pour l'entrée dans le problème G.N. Cependant beaucoup d'élèves s'avèrent incapables de tirer profit des stratégies dont ils disposent. Globalement 55% des enfants ne peuvent pas effectuer de saut qualitatif significatif par rapport à leur première procédure quelle qu'elle soit. Le confinement le plus important a lieu pour les enfants n'arrivant pas à entrer véritablement dans le problème mais aussi pour ceux dont l'activité se concentre sur la réalisation d'un sous-but cohérent avec le problème mais ne prenant pas en compte la totalité de la tâche.

PERSPECTIVES

Les remarques faites précédemment invitent à réfléchir plus particulièrement sur les points suivants:

-Existe-t-il des manières de présenter ces problèmes qui fassent que les procédures intermédiaires utiles à certains enfants constituent de véritables points d'appui pour progresser dans la résolution. Quel rôle joue dans cette dynamique les contrôles que l'enfant peut exercer sur ses propres essais et ses propres procédures ? Est-il possible de les provoquer ou de les inclure comme composante propre de la tâche ?

-L'analyse prédédente offre peu de moyens pour prévoir les sauts qualitatifs importants que certains enfants sont capables de réaliser à partir de certaines séquences d'attente. Quelles sont les compétences spécifiques mises en oeuvre par ces élèves ? Est-il possible de se livrer à un travail spécifique les concernant dans le cadre scolaire ?

- B. PROBLEM SOLVING -

REPRESENTATION DES PROBLEMES PAR LE MAITRE ET PAR L'ELEVE A L'ECOLE ELEMENTAIRE

(Perception of problems by teacher and
pupil in elementary school)

R. BRISSIAUD, J.P. MOREAU, G. PERROT, D. VALENTIN, J. VAUDAY

The aim of this study is to research teachers' and pupils' perception of what a problem is in elementary school: what is the purpose of a problem, what kind of activity is expected, what is highlighted.

The working hypothesis is that the pupil's perception is to a certain extend modelled on the teacher's, and that it models the pupil's study activity and thus determines in part the type of solution procedure that is likely to be employed.

A questionnaire was offered to elementary school teachers. Various opinions concerning the problems are put forward in this questionnaire; the teachers were asked to indicate the degree to which they agreed with each opinion.

The pupils were also offered a questionnaire, in which the child is questioned on the type of problem he prefers(easy or difficult, solved individually or in group), on the importance which he gives to marking, and on his attitude to errors.

We have selected a certain number of answer sheets which constitute profiles of the pupils'perception of problems. The distribution of this profiles has been composed from one class to another in relation to the position of the class teacher on a scale of attitudes which was generated from the teachers' questionnaire: quite a clear dependance is evident. Thus the child's perception is quite strongly modelled by the teacher's perception.It now remains to be seen to what degree this perception determines the modes of study and the solution procedures implemented by the pupil. This will be done by examining wether the distribution of solution procedures varies with the positions of the teachers on the attitude scale. For this purpose the data from the study " Procedures for solving a problem with two unknown factors" will be used.

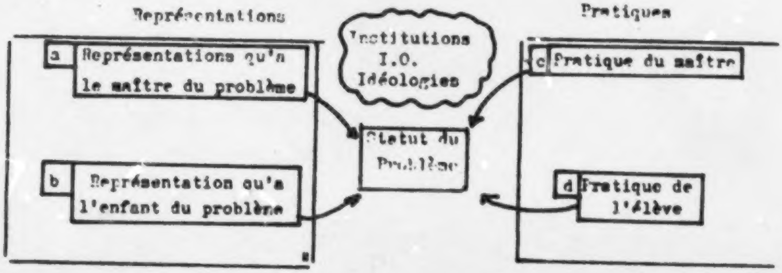
1. - Lorsqu'on tente d'étudier les procédures employées par les enfants lors de la résolution d'un problème qui ne fait pas partie de la "routine" de la classe (c'est-à-dire pour lequel il n'a pas été entraîné), on peut se demander quels sont les facteurs dont elles dépendent. Ceux-ci sont certainement nombreux et de divers ordres, probablement difficiles à mettre en évidenc

Il nous semble que certains sont liés à une pratique antérieure de l'enfant :
- habitude qu'il a d'être mis en situation de recherche
- degré d'initiative qui lui est laissé dans ce cas
- sanctions diverses auxquelles sa recherche est soumise (note, récompense, approbation, etc ...)
- mise en contact (à l'école ou à la maison) avec des jeux et des casse-têtes.

A partir de cette pratique, l'enfant se constitue une représentation du probl qui peut varier dans de très larges proportions. Le mot "problème" peut évoqu
- un plaisir ou un cauchemar
- une activité libre ou très contraignante (nécessité de "rédiger" de telle e telle façon, en tout cas "conforme" au désir du maître)
- un contrôle d'acquisition ou non
- etc ...

Ces représentations de l'enfant, en tant qu'élève d'une classe donnée, sont elles-mêmes en relation avec celles du maître (mais aussi sans doute de ses parents, frères et soeurs, etc ...) nées de sa pratique du problème depuis sa propre enfance, en tant qu'élève et en tant qu'enseignant.

Nous avons appelé (par commodité) "statut du problème" l'ensemble des représentations qu'ont le maître ou l'élève du problème ainsi que les pratiques qui en sont à la fois les causes et les effets.


Représentations
Institutions
I.O.
Idéologies
Pratiques
a
Représentations qu'a le maître du problème
c
Pratique du maître
Statut du Problème
b
Représentation qu'a l'enfant du problème
d
Pratique de l'élève

Nous avons essayé de trouver des moyens nous permettant de mettre en évidence ce "statut".

II. - Les "outils" élaborés

Notre ambition a été de mettre au point des outils se prêtant à l'analyse de chacun des points $\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{d}$, ainsi qu'à leur mise en regard, à leur croisement.

Dans un premier temps nous présenterons les quatre outils, puis nous exposerons l'utilisation que nous en avons faite, ainsi que les résultats obtenus.

a Représentations qu'a le maître du problème

Après diverses tentatives nous avons construit un questionnaire à l'intention des maîtres.

Il a été élaboré à partir d'une série d'entretiens de type semi-directif. Ces entretiens commençaient par la question "Pour vous, qu'est-ce qu'une activité mathématique ?". Ne parlant pas nous-même du problème, nous souhaitions découvrir, dans le corpus de ces entretiens, la place qui lui était réservée. Si, après un certain temps, le maître interrogé n'en parlait pas, la question était abordée de façon explicite : "et les problèmes ?"

Une analyse du contenu et le travail entrepris dans la recherche INRP "Eva - Problèmes" nous a permis de construire le questionnaire maître actuellement utilisé (Document I). Il comprend 23 affirmations extraites des entretiens ou d'"Eva - Problèmes" sur lesquelles on demande aux maîtres de se prononcer en indiquant s'ils sont tout-à-fait, plutôt, plutôt pas, pas du tout d'accord, avec ces assertions.

b Représentation qu'a l'enfant du problème

Un questionnaire à l'intention des élèves a été mis au point (Document II) au sein duquel les renseignements fournis se recoupent. Ainsi les questions 1-7-1 concernent la place du problème dans l'ensemble de l'activité mathématique.

Les questions 2-3-5-8-9 indiquent la fonction du problème dans la classe (prob me de contrôle, de motivation, etc ...).

Les questions 4-6-10 se rapportent surtout aux mises en oeuvre.

C Pratique du maître

Différents moyens d'observation de la pratique des maîtres ont été utilisés : observation de conduite de classe ; étude du cahier journal et des cahiers des enfants, et, plus spécialement, étude de la procédure utilisée pour la correcti des problèmes

Nous avons émis l'hypothèse que lors des séances de correction et d'évaluation, le maître met l'accent sur ce qui lui paraît le plus important. En ce sens, observer une correction devrait permettre d'obtenir un certain éclairage sur les fonctions que le maître attribue aux problèmes, ce qui lui paraît essentiel etc ...

Après avoir pensé qu'il serait préférable de laisser le choix du problème au maître ce qui fournirait des renseignements complémentaires : tout en rendant la situation moins contraignante, nous avons décidé d'imposer les problèmes de façon à pouvoir plus facilement comparer les types de correction, d'une clas à l'autre.

Les problèmes retenus sont des problèmes numériques classiques figurant dans l'enquête sur "l'enseignement mathématique à l'école élémentaire".

Nous avons demandé aux maîtres de faire faire les problèmes individuellement et sur feuille avec un objectif de contrôle : -de corriger les productions des élèves (avant la correction collective si possible) -de nous laisser assister à la séquence de correction dans la classe.

Après avoir observé et analysé plusieurs séquences de correction (dans 24 class nous avons construit des grilles d'analyse.

Il apparaît que cet outil permet d'obtenir un bon éclairage sur les pratiques du maître ; en particulier sur la multiplicité des approches d'une même questio et du développement de l'auto-contrôle.

Malheureusement beaucoup de maîtres se montrent réticents, peut être parce que cette activité ne correspond pas vraiment à leur pratique habituelle, peut être aussi parce qu'ils sentent bien tout ce qu'elle révèle ...

d Pratique de l'élève

Afin d'étudier les procédures et le niveau de compétence des élèves, nous les avons soumis d'une part à des passations collectives sur problèmes.étalonnés au niveau national, et d'autre part à des passations individuelles, dans des conditions normalisées, sur le problème de la Tire-lire (cf Exposé Procédures).

Nous avons voulu que toutes ces approches se recoupent et se contrôlent les unes les autres.

Elles voudraient permettre ne serait-ce qu'une ébauche d'explication des procédures employées par les enfants, en situation de résolution d'un problème, à parti de son "histoire" de petit chercheur et de celle de ses maîtres.

Nous avons décidé d'un ordre de passation des différentes épreuves

1) Questionnaire - enfants outil b)
2) Observation des procédures outil d)
3) Questionnaire - maître outil a)
4) Observation d'une correction outil c)

Enfin, cette investigation nécessite l'adhésion des maîtres. On peut penser que ceux qui acceptent de passer à travers toute cette "course d'obstacles" ne sont peut être pas très représentatifs de l'instituteur français ... Qu'en est-il de leurs élèves ?

IV. - Utilisation des outils et premiers résultats

En préalable, il nous parait important de dire qu'au jour où ce texte est rédig nous n'avons effectué qu'une analyse "à la main" de l'information collectée. Lors de la présentation, à Anvers, de nos résultats, le traitement informatique et statistique des données, par les méthodes classiques, nous permettra de détailler beaucoup plus.

Actuellement nous nous contentons de décrire nos directions de travail, et les tendances générales qui apparaissent de manière suffisamment fortes pour ne pas risquer d'être invalidées ultérieurement.

L'analyse portera successivement sur les résultats apportés par chacun des quatre outils, puis sur certains croisements.

1) Questionnaire maître

Notre méthode a consisté, au vu des deux cents questionnaires recueillis, à extraire des onze questions celles qui discriminaient le mieux, afin d'établir des profils de maitres. Nous avons ainsi constitué, à partir d'une technique de type matrices de Bertin, quatre groupes de maîtres : les deux groupes extême désignés par A et D, correspondent grossièrement - pour les premiers aux maîtr considérant les problèmes comme un outil de construction des notions mathéma tiques, - pour les seconds aux maîtres considérent les problèmes comme un moyen de vérifier l'acquis de ces notions au travers de leur réinvestissemen dans la résolution.

Les deux groupes intermédiaires sont bien sûr moins aisément interprétable leur discours étant moins tranché, voire contradictoire.

2) Représentations qu'a l'élève du problème

A ce jour nous avons étudié 1414 questionnaires, et nous avons là encore cherché à définir des patrons de réponse en tenant compte d'une part de la lettre des réponses, et d'autre part d'une variable qui s'est avérée très signifiante, à savoir la relation au questionnaire : nous avons pu nous rend compte que suivant les classes, les enfants s'autorisent plus ou moins à ne pas répondre à certaines questions, ou à utiliser des réponses nuancées.

A ce jour nous avons constitué une série de sept patrons de réponses, réalisant une partition des protocoles, ce qui nous a permis de situer chaque classe, en fonction de la répartition moyenne au sein des sept patrons de réponses.

3) Pratique du maître

Les résultats obtenus nous paraissant trop biaisés par rapport à la populati nationale, compte tenu de la réserve des maîtres face au regard extérieur, a fortiori, lorsque ce regard apparait comme évaluateur, il ne nous semble pas possible actuellement de les communiquer.

4) Pratique de l'élève

cf Exposé Procédures

5) Croisements

Trois des quatre moyens d'investigation ayant été retenus, deux croisements nous ont paru particulièrement intéressants ;

a) Comparaison entre les représentations du maître et de l'élève

Il apparait nettement, dans les réponses aux questionnaires élèves, un "effet-classe", c'est-à-dire que peu de classes répondent de manière dispers par rapport à chacun des sept protocoles. Pour chaque classe, un ou deux patrons sont surreprésentés massivement. En outre, on constate que, lorsque le maître est classé en catégorie A ou D, à partir de ses réponses au questionnaire maître, on peut assez facilement prévoir la tendance de réponse des élèves, ce qui n'est pas le cas avec les maîtres de catégorie B ou C. Tout semble se passer comme si ces maîtres, étant moins "lisibles", pour leurs élèves comme pour nous, avaient des classes moins monolithiques, à opinion plus hétérogènes.

b) Comparaison entre pratique de l'élève et représentations du problème, che le maître ou l'élève

Ce travail est actuellement en cours de traitement. Nous espérons, d'ici Jui let, pouvoir fournir des résultats numériques concluants, permettant de savo s'il ya un type de maître (au sens des représentations qu'il affiche du problème) qui favorise plus que les autres les procédures d'essai, de changemen de stratégie et inversement si un autre type de maître n'inhiberait pas ces qualités, mettant ainsi leurs élèves dans des situations très différentes pa rapport à l'apprentissage des mathématiques.

THE TEACHER AND DIAGNOSIS: TOO MUCH PIAGET?

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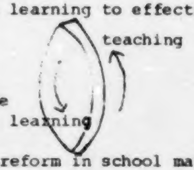
Learning and teaching interact powerfully. Inadequate learning can be due to inadequate teaching, effective learning to effective teaching.

Diagnosis of learners' misconceptions is therefore only one side of the coin.



Awareness by teachers of learners' misconceptions is the other side.

The sixties saw the growth of reform in school mathematics teaching and the development of some major curriculum projects. The seventies was partially a decade of disillusionment internationally and witnessed the growth of diagnostic testing. In the eighties perhaps we shall effectively cement the knowledge and experience gained from the two previous decades.

In the U.K. beginning in 1970 a series of diagnostic studies in schools and continuing education was pioneered at Brunel. These studies broke new ground in diagnostic techniques both statistically and in 'talking through the solution route' aloud. In the mid-seventies the APU began its work on national sampling in mathematics of 11 and 15 year old pupils and together with the CSMS work at Chelsea gave more diagnostic detail of school mathematics. Those of us involved in work of this kind hope that it has helped teachers gain a clearer picture of pupils' thinking.

We are possibly in danger of concentrating our attention exclusively on the learner. If we ignore the effect of the teacher then we introduce a bias to the detriment of the learner. Many of the studies relating to what learners can and cannot do have been based on the Piaget model- hence the sub-title! One side of the coin is well polished!

Work has been done by several of us in the U.K. on teaching strategies but for these to be truly effective teachers must first have the opportunity to be made really aware of the real nature of their learners' deficiencies. The following is a quotation taken from the HMI report "Aspects of Secondary Education in England" (1):

"There is real concern about the lack of basic skills in arithmetic but efforts to remedy this take the form of more and more sets of questions on fractions and decimals without identifying the basic number deficiencies that are at the heart of the problem" (writer's underlining).

There is now a relative wealth of information and resource available to teachers: what is now needed is a diagnostic outcome. The discussion in this paper is concerned essentially with such an outcome.

In this paper I wish therefore briefly to describe our work at Brunel. The studies are represented in figure 1; the aim is an artistic one as a contrast to the usual mathematical model!

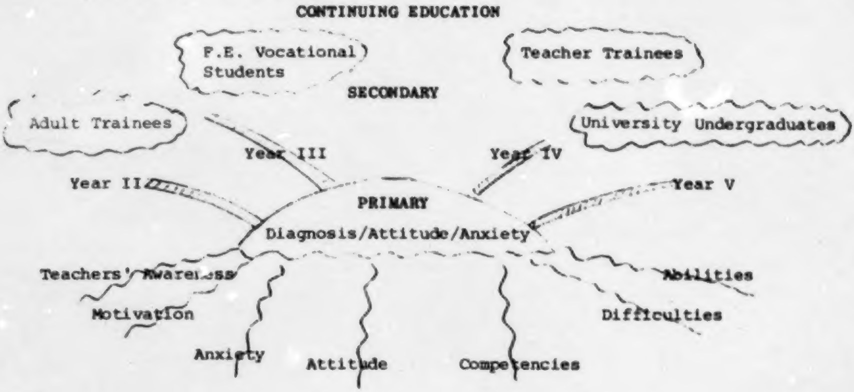

CONTINUING EDUCATION
F.E. Vocational Students
Teacher Trainees
SECONDARY
Adult Trainees
University Undergraduates
Year III
Year IV
Year II
Year V
PRIMARY
Diagnosis/Attitude/Anxiety
Teachers' Awareness
Abilities
Motivation
Difficulties
Anxiety
Attitude
Competencies

Figure 1 THE "TREE" OF LEARNERS

These studies involved a total of approximately 22 000 learners and some of their teachers. The research method is illustrated in Figure 2.

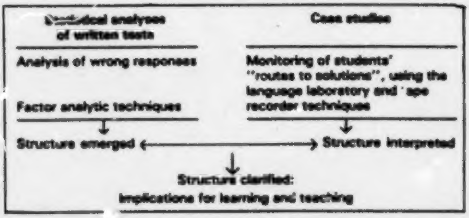

Statistical analyses of written tests
Case studies
Analysis of wrong responses
Factor analytic techniques
Monitoring of students' "routes to solutions", using the language laboratory and tape recorder techniques
Structure emerged
Structure interpreted
Structure clarified:
Implications for learning and teaching

Fig. 2 Complementary research methods.

Some of these studies have been reported at other PME conferences (2).

There are arising from these studies 3 main points for discussion:

A) What is learnt at school does matter for later learning, work and training.

B) Teachers' awareness of learners' misconceptions is critical for effective and efficient teaching.

C) A general structure as a diagnostic outcome to which teachers can relate is essential for efficient teaching.

A) The Learning Spiral. In the U.K. there is concern over soaring unemployment. The Manpower Services Commission with government resources has been asked to initiate training schemes for school leavers and older people. One of our current studies is concerned with mathematical diagnosis for adult trainees together with appropriate teaching packages. Another study recently completed with some Schools Council support has been teachers' use of diagnostic material in the classroom with their 13 year old pupils.

My colleague George Barr will be reporting on our various studies including these two and comparing diagnostic results at various levels. Here is just a taster! Table 1 compares performance of 13 year old pupils with adult trainees of age range 19-45 years.

	13 year old		Adults	
	F	W	F	W
$\frac{2}{x} = \frac{1}{3}$	82%		78%	
$\frac{1}{x} = \frac{3}{4}$	10%	25% $(\frac{3}{4})$	21%	28% $(\frac{3}{4})$

Table 1 F is Facility Value; W is popular error

Another of our studies shows that the learning of more complex mathematics even with university undergraduates continues to be undermined by their misconceptions in elementary mathematics.

B) Teachers' Awareness of Learners' Misconceptions.

Table 2 shows the teachers' expectations of their pupils' performance on the tasks shown in Table 1.

	13 year old F	Teachers' Expectations
$\frac{2}{x} = \frac{1}{3}$	82%	52%
$\frac{1}{x} = \frac{3}{4}$	10%	33%

Table 2

Tables 1 and 2 show that two tasks can look similar with respect to the mathematical processing involved and yet learners will see them quite differently from the teachers - one task diagnoses the misconception, the other does not.

Figures 3 and 4 show the more general results of experienced teachers' expectations in studies carried out 10 years apart, in 1971 and 1981 respectively.

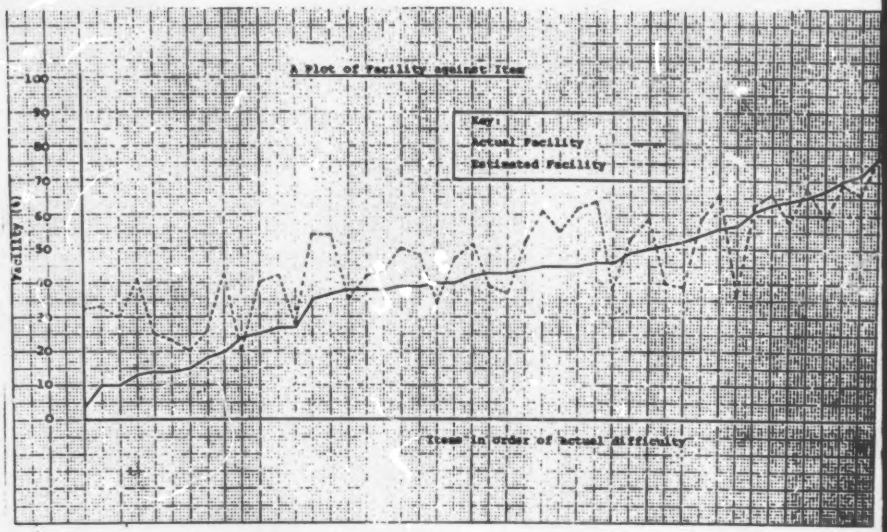


Figure 3 F.E. Lecturers' Expectations of Students' Performance (1971)

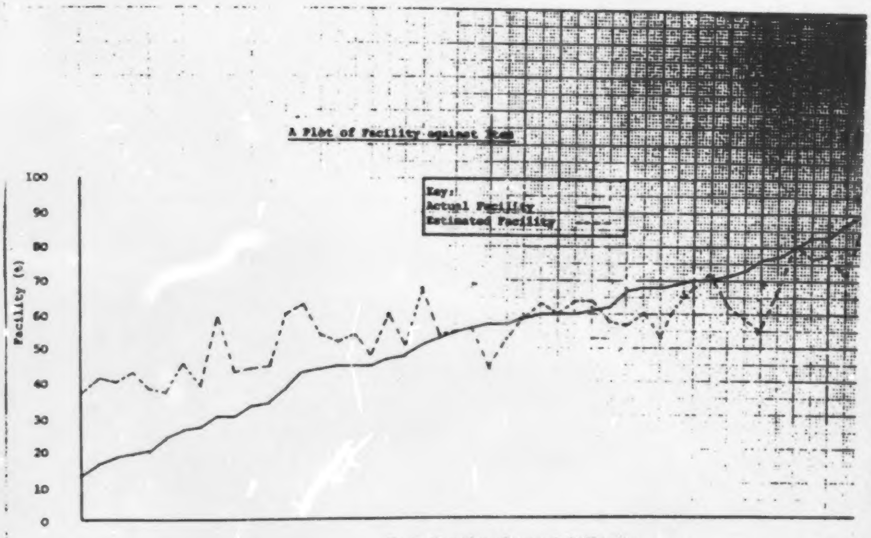


Figure 4 School Teachers' Expectations of Pupils' Performance (1981)

It can be seen from figures 3 and 4 that there were some tasks which learners can do better than teachers expect and some tasks which learners perform significantly worse than teachers expect. For effective teaching the teacher must be aware of the real nature of the misconception. For efficient teaching the teacher must be able to balance teaching time appropriately.

In both studies the significant underestimation of difficulty by teachers in F.E. and school sectors related to certain tasks, called 'inference' tasks; which diagnosed the misconceptions of the school pupils and older students. Another study (3) involved teachers in training at the end of their mathematics course. It was found that these same misconceptions were possessed by these trainees, especially those going to teach in primary school. For example, on a task similar to $\frac{1}{x} = \frac{3}{4}$, the facility value was 34% whilst on that relating to the area of a circle, it was 17% and the errors were the same as those of the pupils.

A learning-teaching cycle is thus indicated - misconceptions which arise in primary school can spiral into secondary and into later life as shown.

C) A General Structure From A Diagnostic Outcome. Our studies indicate firstly that learners' performance in mathematics is determined by their general ability. Secondly, learners' performance also depends on what appears to be a more specific mathematical ability related to the diagnostic/inference tasks (2(iii)). Figure 5 illustrates some of the tasks which characterize these two types of ability.

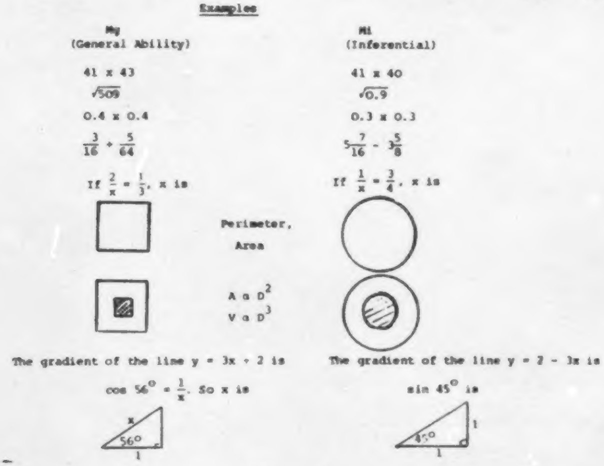

Examples
Mg
(General Ability)
Mi
(Inferential)
41 x 43
41 x 40
$\sqrt{509}$
$\sqrt{0.9}$
0.4 x 0.4
0.3 x 0.3
$\frac{3}{16} + \frac{5}{64}$
$5\frac{7}{16} - 3\frac{5}{8}$
If $\frac{2}{x} = \frac{1}{3}$, x is
If $\frac{1}{x} = \frac{3}{4}$, x is
Perimeter,
Area
$A \alpha D^2$
$V \alpha D^3$
The gradient of the line y = 3x + 2 is
The gradient of the line y = 2 - 3x is
cos 56° = $\frac{1}{x}$. So x is
sin 45° is
x
56°
1
45°
1
1

Figure 5

Figure 5 emphasizes the subtlety of diagnosis in that of tasks which appear to be similar one will diagnose the nature of the misconceptions whilst another will not.

Some characteristics of these two categories of tasks are summarized in Figure 6 where they are linked to Richard Skemp's relational and instrumental understanding of mathematics as an intelligent activity.

Mathematics As An Intelligent Activity	
Mg	Mi
Speed,Accuracy: Routine/ Rote/Algorithmic Application	Reflective/Inferential/Adaptable/ Relational/Intuitive
i.e. Instrumental Learning	i.e. Relational Learning

Figure 6

Conclusion Teachers now have access to a variety of results and discussion on the diagnosis of learners' misconceptions. However, concentration on the learner is only one face of the coin.

Teacher awareness is a critical factor in the learning process and is the other face of the coin. Our studies suggest strongly that explicit teaching of the concepts underlying the diagnostic tasks does result in more effective learning. Relational learning is linked to relational teaching as Richard Skemp stresses.

The following quotation is a typical comment from the teachers with whom we have worked.

"Having been made aware of various basic concepts I found that I was teaching in a different way, not just with the third year but with all years including a fourth year 'O'-level set. I became aware that I was assuming too much previous learning and understanding.

Awareness linked to a general theoretical structure can help us as teachers to be more constructive in our teaching. We need to marshall all that is available in order to construct our own relevance for the classroom.

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LEARNERS, TEACHERS AND MATHEMATICS

A Selection Of Some Results Of Brunel Studies From 10+ To 27+.

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Introduction

The overall aim of this contribution will be to discuss certain concepts and skills (tasks), tracing the learners' performance from primary school (10+) to continuing education and training (27+). This discussion will be based on a group of tasks and developed with reference to data concerned with the learners' performance and teachers' expectations of their learners' performance.

The main theme of the discussion will be on some general characteristics of these tasks in terms of: a) mathematical content, b) learner performance, c) teacher expectation of performance, d) their diagnostic effectiveness, e) their relationship to the structure of mathematical ability.

The Studies and Samples

The samples in the studies were from a

(i) fourth year primary school background; 10/11 year old mixed ability,
(ii) second year secondary school background; 12/13 year old mixed ability,
(iii) third year secondary school background; 13/14 year old mixed ability,
(iv) fourth/fifth year secondary school background; 15/16 year old above average ability,
(v) TOPS trainee intake background; 18/55 year old mixed ability.

The samples from the first two studies answered an open ended power test, while the samples from the latter three studies answered a multiple-choice power test. All the tests had certain items/tasks or themes in common. The aims, objectives and results of these studies are documented by Curnyn (1979,80), Barr (1981), Rees (1978) and Barr (1982) respectively. The overall ethos of these studies has involved an investigation of common learning difficulties experienced by students and how these relate to the structure of mathematical ability (for summary see Rees (1981)).

The Tasks and Facilities

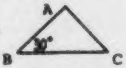
We shall consider 8 of these common tasks. The performance for all the samples on these tasks can be seen in table 1.

The Development of the Concepts

It will be seen that there is a general trend of increasing facility with age for all tasks except task 7 and that the popular common wrong responses are generally the same for all age groups.

Consider tasks 1,2 and 3. We can see from the facilities of tasks 1 and 2 that the concepts of 'long multiplication' and 'long division' develop to a significant

Table 1

Tasks	Primary 4th Year Answer (%)	Popular Error (%)	Secondary 2nd Year Answer (%)	Popular Error (%)	Secondary 3rd Year Answer (%)	Popular Error (%)	Secondary 4/5th Year Answer (%)	Popular Error (%)	TOPS Trainees Answer (%)	Popular Error (%)
1. 41 x 40	1640(53)	164(4)			1640(89)	164(3)	1640(88)	164(5	1640(90)	164(4)
2. 513 ÷ 27	19(32)	9(3)	19(43)	17(1)	19(70)	17(7)	19(70)	17(16)	19(73)	17(5)
3. $23\overline{)4669}$					203(46)	23(40)			203(39)	23(49)
4. 0.3 x 0.3	0.09(9)	0.9(47)	0.09(8)	0.9(56)	0.09(42)	0.9(47)	0.09(51)	0.9(35)	0.09(37)	0.9(57)
5. 0.7 ÷ 0.7	1(49)	0.1(24)	1(32)	0.1(24)	1(35)	0.1(42)	1(62)	0.1(24)	1(27)	0.1(41)
6. $\frac{2}{\square} = \frac{1}{3}$ What number goes in the box?	6(54)	4(13) 3(4	6(70)	3(4)						
If $\frac{2}{x} = \frac{1}{3}$ then x is?					6(82)	$\frac{2}{3}$(5) 3(4	6(87)	3(6) $\frac{2}{3}$(6)	6(78)	3(4) $\frac{2}{3}$(4)
7. If $\frac{1}{x} = \frac{3}{4}$, then x is.					$\frac{4}{3}$(10)	$\frac{3}{4}$(25)	$\frac{4}{3}$(43)	$\frac{3}{4}$(31)	$\frac{4}{3}$(21	$\frac{3}{4}$(28)
8.  If side AB is equal to side AC then the angle at A is?	120°(24)	60°(18)	120°(46)	60°(11)	120°(65)	60°(11)	120°(70)	60°(8)	120°(53)	60°(17)

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competency by the age of 13/14. However, the performance on task 3 suggests that the concept of 'long division' needs 'fine tuning'; this task clearly highlights a difficulty. This difficulty involves a zero. One can see that the popular wrong response for task 1 also involves a zero, but that the difficulty is not so marked for this task compared with task 3.

Tasks 4 and 5 deal with the solution of the 'one over x' problem. It is interesting to note that task 4 is answered correctly by a large fraction of the sample in older age groups while task 5 is found to be difficult. Is this because task 4 has an easy intuitive solution and that this ability to intuit breaks down for task 5?

Tasks 6 and 7 deal with operations on small numbers. The performance on these tasks suggests that these concepts do not develop as well as the other concepts examined. In fact the performance on task 7 appears to deteriorate.

Task 8 shows a consistent improvement in performance, which suggests that this concept from geometry develops from primary to secondary third year and is maintained into adulthood.

The Similarities

It will be seen from table 1 that the popular wrong response for the primary fourth year pupil is likely to be the popular wrong response for the TOPS trainee in continuing education. Further the fraction of the sample giving the popular wrong answer is similar in magnitude from the youngest to the oldest age group.

It should also be noted that for tasks 3,4,5 and 7 the percentage of the sample giving the popular wrong response is often greater than the percent of the sample giving the correct response. It is these items that will now be discussed with the addition of further data.

Teacher Expectations

In a recent study on the use of diagnostic assessment in the classroom (see Barr (1981)) teachers were asked to estimate their pupils' performance and predict the popular error for the tasks in a test. Table 2 gives the estimated facility given by the teachers compared with the actual facility for the 13/14 year old pupils in the sample already discussed.

Tasks	Estimated Facility (E%)	Actual Facility (F%)	Difference in Facility (E%-F%)
3. $23\overline{)4669}$	64	46	18
4. 0.3×0.3	51	42	9
5. $0.7 \div 0.7$	54	35	19
7. $\frac{1}{x} = \frac{3}{4}$	33	10	23

Table 2

Table 2 suggests that for these items the teachers' expectations of performance is optimistic, since the expected facility for these tasks is always greater than the actual facility. One conclusion from this is that the difficulty of certain tasks is sometimes greater than we, as teachers, perceive.

The teachers correctly predicted the popular wrong responses for tasks 3,4 and 5 which suggests that the teachers were aware of the type of error pupils are likely to make but were not aware of the fraction of their pupils likely to make such an error.

In this experimental study on the use of diagnostic assessment the teachers were asked to mark their pupils' test scripts and compare pupil performance with their expectation of pupil performance. The teachers felt this exercise made them aware of the extent and nature of their pupils' difficulties.

One of the teachers in this study noted: "I became aware (during the period of the study) that I was assuming too much previous learning and understanding and I feel I now have a better idea of when and how to question a students' lack of understanding".

Three months after the pupils had first been tested they were once again tested. The pupil performance on items 3, 4, 5 and 7 had increased by 9%, 15% 8% and 8% respectively. These gains must be seen in the light of the fact that the teachers did not necessarily concentrate specifically, in terms of teaching time, on the concepts that the tasks tested.

Diagnostic Tasks

This brief examination of eight tasks that were common to the five age groups shows that certain tasks (i.e. tasks 1,2,4,6 and 8) give us a glimpse of how concepts appear to develop while other tasks (i.e. tasks 3,5 and 7) give us insight into student misconceptions. It is these latter tasks which are diagnostic since they seem to probe understanding and reveal blockages not simply for the primary sample but for all the samples even the above average 4/5th year secondary sample.

Some of the tasks may be matched, (e.g. tasks 2 and 3, 6 and 7). If we consider tasks 2 and 3 they "on the surface" appear to be testing the same concept, of long division, in a similar way, yet performance on these tasks is quite different. Table 3 gives teacher expectations of pupil performance compared with actual performance for tasks 2 and 3. This table highlights how two similar tasks test understanding at different levels, but that this may not be obvious without the pupils attempting the tasks. The ability to estimate the difficulty of a task is a skill which we as teachers clearly need to develop if we are to be effective in the classroom.

Tasks	Estimated Facility E%	Actual Facility F%
2. $513 \div 27$	69	71
3. $23\overline{)4669}$	64	46

Table 3

Conclusion From the discussion of these tasks we can see that depending on the questions we ask, we shall test different levels of understanding; only certain questions will locate the nature of student misconceptions. As teachers we need to become skilled at asking the right questions which probe understanding. By asking the right questions we become more aware of pupil difficulties and so more able to help them develop their mathematical ability.

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